Available online at www.sciencedirect.com

SCIENCE DIRECT®

International Journal of Solids and Structures 41 (2004) 7009–7033

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

Restrictions on linear constitutive equations for a rigid heat conducting Cosserat shell

M.B. Rubin *

Faculty of Mechanical Engineering, Technion, Israel Institute of Technology, Technion city, 32000 Haifa, Israel

Received 1 December 2003
Available online 6 July 2004

Abstract

Although the three-dimensional field equation for linear heat conduction is simple, it is still challenging to obtain solutions of boundary value problems for shells with general geometry. The formulation of such problems can be simplified by using specialized equations which model heat conduction in rigid shells in terms of two temperature fields: one for the average temperature and the other for the average temperature gradient through the shell's thickness. The resulting equations are simpler because the field quantities are independent of the coordinate through the shell's thickness. However, constitutive equations for the heat fluxes in the shell theory are complicated because they depend on both the heat conduction coefficient of the material being considered and on the shell's geometry. The objective of this paper is to develop restrictions on the constitutive equations in the linear Cosserat theory of rigid heat conducting shells which ensure that the Cosserat equations produce exact steady-state solutions for Fourier conduction with an arbitrary constant temperature gradient for all shell geometries including variable thickness. Constitutive equations are proposed which satisfy these restrictions and example problems of a plate and of circular cylindrical and spherical shells are solved which examine the accuracy of the Cosserat theory. The results of these examples show that the Cosserat theory is accurate for moderately thick shells and moderately strong variation of the temperature field through the shell's thickness. In particular, the Cosserat solution converges smoothly to the exact solution as the shell becomes thin. In contrast, the two other theories considered are shown to predict incorrect slopes at the thin shell limit.

Keywords: Cosserat shells; Heat conduction; Rigid

1. Introduction

Classical interest in thermal effects in structures (e.g. Boley and Weiner, 1960; Hetnarski, 1986, 1987, 1989, 1996) has focused mainly on predicting deformations and stresses due to thermal loads and not on predicting heat conduction in the structure. Naghdi (1972) developed general thermomechanical equations for predicting the evolution of both the deformations and the thermal fields within the context of the theory

* Tel.: +97-24-8293188; fax: +97-24-8295711.
E-mail address: mbrubin@tx.technion.ac.il (M.B. Rubin).

0020-7683/$ - see front matter
doi:10.1016/j.ijsolstr.2004.05.031

of a Cosserat surface. This theory was reconsidered in Green and Naghdi (1979) from the new thermodynamical perspective proposed in Green and Naghdi (1977, 1978). Lukasiewicz (1989) also developed a theory for determining thermal stresses in shells and an alternative approach to coupled thermoelastic shell equations which includes the effects of hyperbolic heat conduction can be found in Altay and Dokmeci (2001).

Although the three-dimensional equations of linear heat conduction in rigid solids are relatively simple and many solutions are known (Carslaw and Jaeger, 1956), it is still a challenge to obtain solutions for general shell geometries. Consequently, it is useful to develop specialized equations for heat conduction in rigid shells which introduce simplifications of the temperature field through the shell's thickness. For example, the Cosserat theory (e.g. Naghdi, 1972; Green and Naghdi, 1979) introduces two temperature fields which characterize the average temperature and the average temperature gradient through the shell's thickness. In Rubin (1986) these Cosserat equations were specialized for the case of heat conduction in rigid shells. The simplified theory of Lukasiewicz (1989) also introduces two temperature fields which are determined by energy equations for the shell. More recently, Shvets and Flyachok (1999) have developed a set of equations for multilayer anisotropic shells using a Bubnov–Galerkin approximation of the energy equation. Specifically, they introduced four temperature fields associated with a polynomial approximation up to cubic order in the thickness coordinate. In addition, to further simplify the equations they approximated the local curvature as the curvature on a single reference surface in the shell.

Another application of heat conduction in rigid shell-like structures is related to interphases in composite materials. In the works of Sanchez-Palencia (1970), and Pham Huy and Sanchez-Palencia (1974) the interphase was approximated as a single interface surface and imperfect interface conditions were proposed separately for weakly and highly conducting interphases. Recently, Hashin (2001) used a Taylor series expansion to develop heat conduction equations for thin (but finite thickness) interphases. These equations are shell-type equations which introduce two temperature fields and are valid for interphases with general geometry. In contrast with the previous work which proposed different equations for weakly conducting interphases from those for highly conducting interphases, Hashin's equations are unified in the sense that the same equations are valid for the entire range of conductivity of the interphase.

Within the context of the Cosserat theory for heat conduction in rigid shells, the dependence of the two temperature fields on the coordinate through the shell's thickness is eliminated. This simplifies the equations since the field quantities depend only on time and the two spatial coordinates which characterize material points on the shell's middle surface. However, this elimination procedure causes the constitutive equations for resultant flux quantities to depend on the shell's geometry even when the associated three-dimensional fluxes (e.g. for Fourier heat conduction) are trivial. Consequently, even within the context of linear theory, the development of specific functional forms for these resultant flux quantities is not trivial.

For the purely mechanical theory of shells, it is well known that the constitutive equations for shells depend on both the properties of the material used to manufacture the shell and on the specific geometry of the shell structure. However, given a specific uniform homogeneous elastic material and a specific shell geometry it is not known how to specify the strain energy function for the elastic shell structure. Naghdi and Rubin (1995) made some progress in sorting out the individual contributions of material and geometry properties of the shell by developing restrictions on the constitutive equations for shells which ensure that the shell equations produce exact solutions for all homogeneous deformations of non-linear elastic shells with general reference geometry. These restrictions are fundamental in nature because they are valid for large deformations.

In this paper, attention is confined to Fourier heat conduction but the rigid shell can have general geometry including variable thickness. It is well known that the equilibrium equation for steady-steady heat conduction in the absence of external heat supply requires the divergence of the three-dimensional heat flux vector to vanish. For linear Fourier heat conduction this equation is automatically satisfied for all constant temperature gradients. Moreover, since the Cosserat theory with two temperature fields can model a general constant three-dimensional temperature gradient exactly, it should be possible to develop restrictions on the thermal constitutive equations for shells which ensure consistency with exact solutions for arbitrary constant temperature gradients. The objective of this paper is to develop these restrictions.

In contrast with the purely mechanical restrictions (Naghdi and Rubin, 1995) which were valid for general non-linear homogeneous deformations, these thermal restrictions are valid only for the linear theory of heat conduction with small temperature variations from a uniform reference temperature. This is because for the general non-linear theory of heat conduction the heat flux vector can depend on both the temperature and the temperature gradient. Consequently, the linear spatial variation of the temperature field associated with a constant temperature gradient can cause non-vanishing divergence of the heat flux. However, for the linear theory of Fourier heat conduction these thermal restrictions still impose non-trivial conditions on the constitutive equations since they are valid for general shell geometry. In this regard, it is noted that the constitutive equations in Rubin (1986) do not satisfy these restrictions.

An outline of this paper is as follows: Section 2 briefly reviews equations for a rigid heat conductor which are a special case of the three-dimensional thermodynamical formulation proposed by Green and Naghdi (1977, 1978). Section 3 develops the equations for a heat conducting rigid Cosserat shell by the direct approach. Section 4 develops the restrictions on the constitutive equations which ensure that the linear Cosserat equations produce exact solutions for arbitrary constant temperature gradients in a shell with general geometry. Section 5 summarizes the models proposed by Lukasiewicz (1989) and Hashin (2001). Section 6 considers an example of transient heat conduction in a plate to establish the validity a modified constitutive coefficient. Section 7 specializes the Cosserat equations for a general cylindrical shell, and Section 8 considers example problems for a circular cylindrical shell. Section 9 specializes the Cosserat equations for a spherical shell and Section 10 considers an example problem for a spherical shell. A summary of the main results is presented in Section 11. Finally, Appendix A provides details of the three-dimensional approach.

Throughout the text, the usual summation convention is used for repeated lower case indices with the range of Latin indices being ($i = 1, 2, 3$) and that of Greek indices being ($\alpha = 1, 2$).

2. Three-dimensional theory

The objective of this section is to briefly summarize aspects of the three-dimensional thermodynamic formulation proposed by Green and Naghdi (1977, 1978). Specifically, attention is confined to a rigid heat conductor and initially the absolute temperature field $\theta^*(\theta^* > 0)$ is not restricted to small variations from the uniform constant reference temperature θ_0. Also, to ease comparison of the three-dimensional equations with corresponding equations in the Cosserat theory of shells discussed in the next section, similar quantities appearing in both theories are denoted by the same symbol, but with a superposed (*) attached to the symbol associated with the three-dimensional theory.

In this theory the temperature field is determined by solving a balance of entropy proposed in the form

$$\rho^*\dot{\eta}^* = \rho^* s^* + \rho^* \xi^* - \mathrm{div}^* \mathbf{p}^*, \tag{2.1}$$

where ρ^* is the mass density, η^* is the specific (per unit mass) entropy, s^* is the specific external rate of supply of entropy, ξ^* is the specific internal rate of production of entropy, $\mathbf{p}^*$ is the entropy flux per unit area, and div^* is the divergence operator with respect to the position $\mathbf{x}^*(\theta^i)$ of a material point. In general the quantities depend on three convected coordinates θ^i $(i = 1, 2, 3)$ and time t. Also, the balance of energy is given by

$$\rho^*\dot{\varepsilon}^* = \rho^*\theta^* s^* - \mathrm{div}^*(\theta^*\mathbf{p}^*) = \rho^* r^* - \mathrm{div}^*(\mathbf{q}^*), \tag{2.2}$$

where ε^* is the specific internal energy, and the rate of external supply of energy r^* and the heat flux vector $\mathbf{q}^*$ are related to s^* and $\mathbf{p}^*$ by the expressions

$$r^* = \theta^* s^*, \quad \mathbf{q}^* = \theta^* \mathbf{p}^*. \tag{2.3a,b}$$

Moreover, it can be shown that

$$\text{div}^*(\theta^* \mathbf{p}^*) = \mathbf{p}^* \cdot \mathbf{g}^* + \theta^* \text{div}^* \mathbf{p}^*, \quad \mathbf{g}^* = \partial\theta^*/\partial\mathbf{x}^*, \tag{2.4}$$

where $\mathbf{g}^*$ is the temperature gradient. Then, using (2.3), (2.4) and the definition

$$\psi^* = \varepsilon^* - \theta^* \eta^* \tag{2.5}$$

of the Helmholtz free energy ψ^*, it follows that the energy equation (2.2) can be rewritten in the form

$$\rho^* \theta^* \xi^* = -\mathbf{p}^* \cdot \mathbf{g}^* - \rho^*(\dot{\psi}^* + \eta^* \dot{\theta}^*). \tag{2.6}$$

Now, for a rigid heat conductor the constitutive equations are assumed to take the forms

$$\psi^* = \psi^*(\theta^*), \quad \eta^* = \eta^*(\theta^*), \quad \mathbf{p}^* = \mathbf{p}^*(\theta^*, \mathbf{g}^*), \quad \xi^* = \xi^*(\theta^*, \mathbf{g}^*). \tag{2.7}$$

In the thermodynamic procedures proposed by Green and Naghdi (1977, 1978) the balance of entropy (2.1) is used to determine the temperature field and the reduced form (2.6) of the balance of energy is used to obtain restrictions on constitutive equations. Specifically, (2.6) is assumed to be valid for all thermomechanical processes, which yields the standard restrictions on these constitutive assumptions of the forms

$$\eta^* = -\frac{\partial\psi^*}{\partial\theta^*}, \quad \rho^* \theta^* \xi^* = -\mathbf{p}^* \cdot \mathbf{g}^*. \tag{2.8a,b}$$

Also, one form of the second law of thermodynamics requires heat (or entropy) to flow from hot to cold regions which is equivalent to requiring the rate of internal production of entropy to be non-negative

$$\rho^* \theta^* \xi^* = -\mathbf{p}^* \cdot \mathbf{g}^* \geqslant 0. \tag{2.9}$$

For the discussion of shells in the next section it is necessary to use general curvilinear coordinates. Most often, this leads to the notions of covariant differentiation. However, following the approach used in (Green and Zerna, 1968; Rubin, 2000) the equations can be expressed in alternative forms which require knowledge only of partial differentiation. To this end, let the base vectors $\mathbf{g}_i^*$, their reciprocal vectors $\mathbf{g}^{*i}$, the scalar $g^{*1/2}$ and the metric g^{*ij} be define by the equations

$$\mathbf{g}_i^* = \mathbf{x}_{,i}^*, \quad \mathbf{g}_i^* \cdot \mathbf{g}^{*j} = \delta_i^j, \quad g^{*1/2} = \mathbf{g}_1^* \times \mathbf{g}_2^* \cdot \mathbf{g}_3^* > 0, \quad g^{*ij} = \mathbf{g}^{*i} \cdot \mathbf{g}^{*j}, \tag{2.10}$$

where a comma denotes partial differentiation with respect to θ^i. Then, the gradient and divergence operators have the properties that

$$\mathbf{g}^* = \text{grad}^* \theta^* = \theta_{,i}^* \mathbf{g}^{*i}, \quad g^{*1/2} \text{div}^* \mathbf{p}^* = (g^{*1/2} \mathbf{p}^* \cdot \mathbf{g}^{*i})_{,i}. \tag{2.11a,b}$$

Next, defining m^* and p^{*i} by

$$m^* = \rho^* g^{*1/2}, \quad p^{*i} = g^{*1/2} \mathbf{p}^* \cdot \mathbf{g}^{*i}, \tag{2.12}$$

the balance of entropy, balance of energy and the second law of thermodynamics can be written in the forms, respectively,

$$m^* \dot{\eta}^* = m^* s^* + m^* \xi^* - p_{,i}^{*i}, \quad m^* \dot{\varepsilon}^* = m^* \theta^* s^* - (\theta^* p^{*i})_{,i}, \tag{2.13a,b}$$

$$m^* \theta^* \xi^* = -p^{*i} \theta_{,i}^* \geqslant 0. \tag{2.13c}$$

For the simplest case of a constant specific heat c and Fourier heat conduction

$$\psi^* = c\left[\theta^* - \theta_0 - \theta^* \ln\left\{\frac{\theta^*}{\theta_0}\right\}\right], \quad \eta^* = c\ln\left\{\frac{\theta^*}{\theta_0}\right\}, \quad \varepsilon^* = c(\theta^* - \theta_0), \quad \mathbf{q}^* = -k\mathbf{g}^*, \quad \mathbf{p}^* = -\frac{k}{\theta^*}\mathbf{g}^*, \tag{2.14}$$

where k is the constant heat conduction coefficient. Then, for small values of s^* and for small temperature variations, quadratic terms in $(\theta^* - \theta_0)$ are neglected and these constitutive equations and (2.3a) reduce to

$$\psi^* = -\frac{c}{2\theta_0}(\theta^* - \theta_0)^2, \quad \eta^* = \frac{c}{\theta_0}(\theta^* - \theta_0), \quad \varepsilon^* = c(\theta^* - \theta_0), \quad \mathbf{q}^* = -k\mathbf{g}^*,$$
$$\mathbf{p}^* = -\frac{k}{\theta_0}\mathbf{g}^*, \quad \rho^*\theta^*\xi^* = 0, \quad r^* = \theta_0 s^* \tag{2.15}$$

and the balance of entropy becomes

$$m^*\dot{\eta}^* = m^* s^* - p^{*i}_{,i}. \tag{2.16}$$

It can easily be shown that apart from a constant factor of θ_0 this is the same equation as that due to the balance of energy (2.2). Moreover, in the absence of external rate of entropy supply ($s^* = 0$) (2.16) reduces to the standard equation for linear heat conduction

$$\rho^* c\dot{\theta}^* = k\nabla^{*2}\theta^*, \quad \nabla^{*2}\theta^* = g^{*-1/2}[g^{*1/2}g^{*ij}\theta^*_{,i}]_{,j}, \tag{2.17}$$

where ∇^{*2} is the three-dimensional Laplacian operator.

3. A rigid heat conducting Cosserat shell

The balance laws of the theory of a Cosserat shell can be developed by integrating the three-dimensional equations or they can be developed using the direct approach which postulates them directly. However, the constitutive equations of the shell are always developed within the context of the direct approach. Here, it is convenient to use the notation proposed in Rubin (2000) which differs from that in Naghdi (1972), Green and Naghdi (1979) and Rubin (1986). Specifically, attention is confined to a shell which has variable normal thickness $H(\theta^\alpha)$, and a material point in the shell is the located by the position vector $\mathbf{x}^*$ which admits the representation

$$\mathbf{x}^* = \mathbf{x}^*(\theta^i) = \mathbf{x}(\theta^\alpha) + \theta^3\mathbf{a}_3(\theta^\alpha), \quad -\frac{H(\theta^\alpha)}{2} \leqslant \theta^3 \leqslant \frac{H(\theta^\alpha)}{2}, \tag{3.1}$$

where $\mathbf{x}$ locates material points on the reference middle surface ($\theta^3 = 0$), $\mathbf{a}_3$ is the unit normal to that surface, and the convected coordinates of the surface are denoted by θ^α ($\alpha = 1, 2$). The tangent vectors $\mathbf{a}_\alpha$, their reciprocal vectors $\mathbf{a}^\alpha$, the metric $a^{\alpha\beta}$ and the scalar $a^{1/2}$ associated with this middle surface are defined by

$$\mathbf{a}_\alpha = \mathbf{x}_{,\alpha}, \quad \mathbf{a}_\alpha \cdot \mathbf{a}^\beta = \delta^\beta_\alpha, \quad a^{\alpha\beta} = \mathbf{a}^\alpha \cdot \mathbf{a}^\beta, \quad \mathbf{a}_3 = \frac{\mathbf{a}_1 \times \mathbf{a}_2}{|\mathbf{a}_1 \times \mathbf{a}_2|}, \quad a^{1/2} = \mathbf{a}_1 \times \mathbf{a}_2 \cdot \mathbf{a}_3 > 0, \quad \mathbf{a}^\alpha = a^{\alpha\beta}\mathbf{a}_\beta. \tag{3.2}$$

In addition, the following identity is recorded for later convenience

$$(a^{1/2}\mathbf{a}^\alpha)_{,\alpha} = -a^{1/2}[(\mathbf{a}^\sigma \cdot \mathbf{a}_{3,\sigma})\mathbf{a}_3]. \tag{3.3}$$

Also, the temperature field is assumed to be represented in the form

$$\theta^* = \theta^*(\theta^i, t) = \theta(\theta^\alpha, t) + \theta^3\theta_3(\theta^\alpha, t), \tag{3.4}$$

where $\theta > 0$ is the average absolute temperature and θ_3 is the average temperature gradient through the shell's thickness. (Here, the common symbol θ for temperature is retained, so powers of θ or θ_3 will be indicated using parentheses to avoid confusion with the convected coordinates θ^i.)

For a rigid heat conducting shell these temperature fields are determined by solving the balances of entropy which take the forms

$$m\dot{\eta} = m(s + \xi) - p^{\alpha}_{,\alpha}, \quad m\dot{\eta}^3 = m(s^3 + \xi^3) + p^3 - p^{3\alpha}_{,\alpha}, \tag{3.5}$$

where m is the mass per unit area $\mathrm{d}\theta^1\,\mathrm{d}\theta^2$, η and η^3 are the specific entropies, s and s^3 are the specific external rates of supply of entropy, ξ and ξ^3 are the specific internal rates of production of entropy, p^{α} and $p^{3\alpha}$ are entropy fluxes and p^3 is an intrinsic rate of supply of entropy. Also, the balance of energy is given by

$$m\dot{\varepsilon} = m(\theta s + \theta_3 s^3) - (\theta p^{\alpha} + \theta_3 p^{3\alpha})_{,\alpha}, \tag{3.6}$$

where ε is the specific internal energy. Appendix A presents a derivation of these equations using the Bubnov–Galerkin approach based on weighted integrals of the three-dimensional equations.

Following the thermodynamic procedures for the three-dimensional theory, the specific Helmholtz free energy ψ of the shell is defined by

$$\psi = \varepsilon - \theta\eta - \theta_3\eta^3 \tag{3.7}$$

and the balances of entropy are used to obtain the reduced energy equation in the form

$$m(\theta\xi + \theta_3\xi^3) = -p^{\alpha}\theta_{,\alpha} - p^3\theta_3 - p^{3\alpha}\theta_{3,\alpha} - m(\dot{\psi} + \eta\dot{\theta} + \eta^3\dot{\theta}_3). \tag{3.8}$$

Now, for a rigid heat conducting shell the constitutive equations are assumed to take the forms

$$\begin{aligned} &\psi = \psi(\theta, \theta_3; \mathscr{G}), \quad \eta = \eta(\theta, \theta_3; \mathscr{G}), \quad \eta^3 = \eta^3(\theta, \theta_3; \mathscr{G}), \\ &p^i = p^i(\mathscr{V}), \quad p^{3\alpha} = p^{3\alpha}(\mathscr{V}), \quad \xi = \xi(\mathscr{V}), \quad \xi^3 = \xi^3(\mathscr{V}), \\ &\mathscr{V} = \{\theta, \theta_3, \theta_{,\alpha}, \theta_{3,\alpha}; \mathscr{G}\}, \end{aligned} \tag{3.9}$$

where $\mathscr{G}$ represents the shell's geometry. Again, using procedures similar to those discussed in Section 2 for the three-dimensional theory, the reduced balance of energy (3.8) is assumed to be valid for all thermomechanical processes which yields restrictions on these constitutive assumptions of the forms

$$\eta = -\frac{\mathrm{d}\psi}{\partial\theta}, \quad \eta^3 = -\frac{\mathrm{d}\psi}{\partial\theta_3}, \quad m(\theta\xi + \theta_3\xi^3) = -p^{\alpha}\theta_{,\alpha} - p^3\theta_3 - p^{3\alpha}\theta_{3,\alpha}. \tag{3.10}$$

Also, one form of the second law of thermodynamics requires heat (or entropy) to flow from hot to cold regions which is equivalent to requiring the rate of internal production of entropy to be non-negative

$$m(\theta\xi + \theta_3\xi^3) = -p^{\alpha}\theta_{,\alpha} - p^3\theta_3 - p^{3\alpha}\theta_{3,\alpha} \geqslant 0. \tag{3.11}$$

For small temperature variations from the constant uniform temperature θ_0 (with $\theta_3 = 0$), the constitutive equation (2.14) in Rubin (1986) suggest that

$$\begin{aligned} &\psi = -\frac{c}{2\theta_0}(\theta - \theta_0)^2 - \frac{cH^2}{2\pi^2\theta_0}(\theta_3)^2, \quad \eta = \frac{c}{\theta_0}(\theta - \theta_0), \quad \eta^3 = \frac{cH^2}{\pi^2\theta_0}\theta_3, \\ &p^{\alpha} = -\frac{kH}{\theta_0}[a^{1/2}a^{\alpha\beta}]\theta_{,\beta}, \quad p^{3\alpha} = -\frac{kH^3}{12\theta_0}[a^{1/2}a^{\alpha\beta}]\theta_{3,\beta}, \\ &\varepsilon = c(\theta - \theta_0), \quad \xi = 0, \quad \xi^3 = 0. \end{aligned} \tag{3.12}$$

These constitutive equations for $\{p^\alpha, p^{3\alpha}\}$ and the constitutive equation (A.10c) for p^3

$$p^3 = -\frac{kH}{\theta_0} a^{1/2} \theta_3, \tag{3.13}$$

for p^3 are consistent with the Bubnov–Galerkin forms (A.10) for plates ($\mathbf{a}_{3,\alpha} = 0$) and can be proposed for shells as well. However, in (Rubin, 1986) p^3 (which is equivalent to $m\xi_1$ there) was specified by the modified form

$$p^3 = -\frac{kH}{\theta_0}\left[a^{1/2} + \frac{H^2}{12}(\mathbf{a}_{3,1} \times \mathbf{a}_{3,2} \cdot \mathbf{a}_3)\right]\theta_3, \tag{3.14}$$

which is consistent with the Bubnov–Galerkin form for general shells. Also, for constant mass density ρ^* the mass term m in the balance laws is given by (A.5). Moreover, it is noted that for shells the explicit dependence of the constitutive equations on the shell's reference geometry was not specified in Green and Naghdi (1979).

Furthermore, it can be shown that the expressions (3.12) for p^α and $p^{3\alpha}$ yield the results

$$p^\alpha_{,\alpha} = -\frac{kH}{\theta_0} a^{1/2} \nabla_s^2 \theta - \frac{k}{\theta_0} H_{,\alpha}\left[a^{1/2} a^{\alpha\beta}\right]\theta_{,\beta}, \quad p^{3\alpha}_{,\alpha} = -\frac{kH^3}{12\theta_0} a^{1/2} \nabla_s^2 \theta_3 - \frac{kH^2}{4\theta_0} H_{,\alpha}\left[a^{1/2} a^{\alpha\beta}\right]\theta_{3,\beta},$$
$$\nabla_s^2 \theta = a^{-1/2}(a^{1/2} a^{\alpha\beta}\theta_{,\alpha})_{,\beta}, \tag{3.15}$$

which express the surface divergences of these quantities in terms of the surface Laplacian ∇_s^2 of the temperature fields θ and θ_3, respectively.

Next, from Appendix A it is recalled that the assigned fields ms and ms^3 can be separated additively into two parts

$$ms = m(s_s + s_p), \quad ms^3 = m(s_s^3 + s_p^3), \tag{3.16}$$

where ms_s and ms_s^3 are due to the three-dimensional external rate of entropy supply s^*, and the terms ms_p and ms_p^3, due to entropy supply through the bottom and top surfaces of the shell, are given by

$$ms_p = -\frac{1}{\theta_0}[\hat{\alpha}\hat{q} - \bar{\alpha}\bar{q}], \quad ms_p^3 = -\frac{H}{2\theta_0}[\hat{\alpha}\hat{q} + \bar{\alpha}\bar{q}],$$
$$\bar{q} = -\bar{\mathbf{q}}^* \cdot \bar{\mathbf{n}}^*, \quad \hat{q} = \hat{\mathbf{q}}^* \cdot \hat{\mathbf{n}}^*, \quad \bar{\mathbf{q}}^* = \mathbf{q}^*\left(\theta^\alpha, -\frac{H}{2}, t\right), \quad \hat{\mathbf{q}}^* = \mathbf{q}^*\left(\theta^\alpha, \frac{H}{2}, t\right) \tag{3.17}$$

for small temperature variations from θ_0. In these expressions $\bar{q}$ is the heat flux into the shell through its bottom surface and $\hat{q}$ is the heat flux out of the shell through its top surface. Furthermore, the unit outward normal vectors $\{\bar{\mathbf{n}}^*, \hat{\mathbf{n}}\}$ to the bottom and top surfaces, respectively, and the scalars $\{\bar{\alpha}, \hat{\alpha}\}$ are given by

$$\bar{\alpha}\bar{\mathbf{n}}^* = -\left[\bar{g}^{*1/2}\bar{\mathbf{g}}^{*3} + \frac{H_{,\sigma}}{2}\bar{g}^{*1/2}\bar{\mathbf{g}}^{*\sigma}\right], \quad \hat{\alpha}\hat{\mathbf{n}}^* = \hat{g}^{*1/2}\hat{\mathbf{g}}^{*3} - \frac{H_{,\sigma}}{2}\hat{g}^{*1/2}\hat{\mathbf{g}}^{*\sigma} \tag{3.18}$$

in terms of the vectors defined in (A.7).

Thus, for small temperature variations the linear forms of the balances of entropy reduce to

$$m\dot{\eta} = ms_s - \frac{1}{\theta_0}[\hat{\alpha}\hat{q} - \bar{\alpha}\bar{q}] - p^\alpha_{,\alpha}, \quad m\dot{\eta}^3 = ms_s^3 - \frac{H}{2\theta_0}[\hat{\alpha}\hat{q} + \bar{\alpha}\bar{q}] + p^3 - p^{3\alpha}_{,\alpha}, \tag{3.19}$$

which are subject to initial and boundary conditions. Now, in the absence of external entropy supply ($s_s = s_s^3 = 0$) and confining attention to steady-state these equations can be solved for $\bar{q}$ and $\hat{q}$ to obtain

$$\bar{q} = \frac{\theta_0}{\bar{\alpha}}\left[\frac{1}{2}p^\alpha_{,\alpha} + \frac{1}{H}(p^3 - p^{3\alpha}_{,\alpha})\right], \quad \hat{q} = -\frac{\theta_0}{\hat{\alpha}}\left[\frac{1}{2}p^\alpha_{,\alpha} - \frac{1}{H}(p^3 - p^{3\alpha}_{,\alpha})\right]. \tag{3.20}$$

Also, using the assumption (3.4) it follows that the temperature fields θ and θ_3 are related to the temperatures $\bar{\theta}$ and $\hat{\theta}$ associated with the shell's bottom and top surfaces, respectively, by the expressions

$$\theta = \frac{1}{2}(\bar{\theta} + \hat{\theta}), \quad \theta_3 = \frac{1}{H}(\hat{\theta} - \bar{\theta}). \tag{3.21}$$

4. Restrictions for constant temperature gradient

For constant temperature gradient $\mathbf{g}^*$ it follows that the three-dimensional temperature field θ^* is given by

$$\theta^* = \theta_A + \mathbf{g}^* \cdot (\mathbf{x}^* - \mathbf{x}_A), \tag{4.1}$$

where θ_A is the temperature at the material point $\mathbf{x}_A$. Moreover, in the absence of external entropy supply and confining attention to steady-state the non-linear balance law (2.1) reduces to

$$\operatorname{div}^*(\mathbf{q}^*) = 0, \tag{4.2}$$

where use has been made of the relationship (2.3b) and the constitutive equation (2.8b). However, in general the non-linear constitutive equation for heat flux depends also on θ^*, which is a linear function of $\mathbf{x}^*$, so that (4.2) is not necessarily satisfied for a constant temperature gradient.

In contrast, for small temperature variations from θ_0, $\mathbf{p}^*$ and $\mathbf{q}^*$ are linear functions of $\mathbf{g}^*$ which are independent of $\mathbf{x}^*$ (since θ^* in $\mathbf{p}^*$ is approximated by θ_0). This means that for steady-state and in the absence of external entropy supply, the approximation of the balance law (2.1) (with $\xi^* = 0$ since it is second-order in temperature) reduces to

$$\operatorname{div}^*\mathbf{p}^* = \frac{1}{\theta_0}\operatorname{div}^*\mathbf{q}^* = 0, \tag{4.3}$$

which is satisfied in general for constant temperature gradient in shells made from uniform homogeneous materials. Consequently, it should be possible to propose restrictions on the constitutive equations for the linear theory of a rigid heat conducting shell which ensure that the resulting steady-state balance laws in the absence of external entropy supply are consistent with the exact equations for arbitrary constant temperature gradient $\mathbf{g}^*$ in shells with general geometry. Specifically, for this case the balances of entropy (3.19) reduce to

$$-\frac{1}{\theta_0}[\hat{\alpha}\hat{q} + \bar{\alpha}\bar{q}] - p^{\alpha}_{,\alpha} = 0, \quad -\frac{H}{2\theta_0}[\hat{\alpha}\hat{q} - \bar{\alpha}\bar{q}] + p^3 - p^{3\alpha}_{,\alpha} = 0. \tag{4.4.a,b}$$

Moreover, substituting (3.1) into (4.1) and using (3.4) yields the expressions

$$\theta = \theta_A + \mathbf{g}^* \cdot (\mathbf{x} - \mathbf{x}_A), \quad \theta_3 = \mathbf{g}^* \cdot \mathbf{a}_3, \quad \theta_{,\alpha} = \mathbf{g}^* \cdot \mathbf{a}_\alpha, \quad \theta_{3,\alpha} = \mathbf{g}^* \cdot \mathbf{a}_{3,\alpha} \tag{4.5}$$

for a constant temperature gradient $\mathbf{g}^*$. Also, in view of the specifications (3.17) and (3.18) it follows that for Fourier heat conduction (2.15), the quantities $\{\bar{\alpha}\bar{q}, \hat{\alpha}\hat{q}\}$ are given by

$$\begin{aligned} \bar{\alpha}\bar{q} &= -k\left[\bar{g}^{*1/2}\bar{\mathbf{g}}^{*3} + \frac{H_{,\sigma}}{2}\bar{g}^{*1/2}\bar{g}^{*\sigma}\right] \cdot \mathbf{g}^*, \\ \hat{\alpha}\hat{q} &= -k\left[\hat{g}^{*1/2}\hat{\mathbf{g}}^{*3} - \frac{H_{,\sigma}}{2}\hat{g}^{*1/2}\hat{g}^{*\sigma}\right] \cdot \mathbf{g}^* \end{aligned} \tag{4.6}$$

in terms of the vectors defined in (A.7).

For general shell geometry Eq. (4.4) are non-trivial even for constant $\mathbf{g}^*$. In particular, using the constitutive equations (3.12) and (3.14) it was shown in Rubin (1986) that these equations were not satisfied for a specific case of constant temperature gradient in a problem for a circular cylindrical shell with constant thickness. The objective here is to modify these constitutive equations so that (4.4) is satisfied for arbitrary $\mathbf{g}^*$ and arbitrary shell geometry. Specifically, it will be shown that the constitutive equation (3.12) for p^α and $p^{3\alpha}$ can be retained but the constitutive equation for p^3 must be modified relative to (3.13) or (3.14).

To this end, Eq. (3.20) can be rearranged to obtain

$$\frac{1}{\theta_0}[\hat{\alpha}\hat{q} - \bar{\alpha}\bar{q}] + p^\alpha_{,\alpha} = 0, \quad p^3 = \frac{H}{2\theta_0}[\hat{\alpha}\hat{q} + \bar{\alpha}\bar{q}] + p^{3\alpha}_{,\alpha}. \tag{4.7}$$

Next, using (3.18), (4.6) and (A.7) these equations yield restrictions on the constitutive equations for $\{p^\alpha, p^3, p^{3\alpha}\}$ of the forms

$$-\frac{k}{\theta_0}[Ha^{1/2}(\mathbf{a}^\sigma \cdot \mathbf{a}_{3,\sigma})\mathbf{a}_3 - H_{,\sigma}a^{1/2}\mathbf{a}^\sigma] \cdot \mathbf{g}^* + p^\alpha_{,\alpha} = 0, \tag{4.8a}$$

$$p^3 = -\frac{k}{\theta_0}\left[H\left\{a^{1/2} + \frac{H^2}{4}(\mathbf{a}_{3,1} \times \mathbf{a}_{3,2} \cdot \mathbf{a}_3)\right\}\mathbf{a}_3 - \frac{H^2}{4}H_{,1}(\mathbf{a}_{3,2} \times \mathbf{a}_3) - \frac{H^2}{4}H_{,2}(\mathbf{a}_3 \times \mathbf{a}_{3,1})\right] \cdot \mathbf{g}^* + p^{3\alpha}_{,\alpha}. \tag{4.8b}$$

Moreover, using (3.3), (4.5) and the constitutive equation (3.12) for p^α and $p^{3\alpha}$ it can be shown that

$$a^{1/2}a^{\alpha\beta}\theta_{,\beta} = a^{1/2}\mathbf{a}^\alpha \cdot \mathbf{g}^*, \tag{4.9a}$$

$$p^\alpha_{,\alpha} = \frac{k}{\theta_0}[Ha^{1/2}(\mathbf{a}^\sigma \cdot \mathbf{a}_{3,\sigma})\mathbf{a}_3 - H_{,\sigma}a^{1/2}\mathbf{a}^\sigma] \cdot \mathbf{g}^*, \tag{4.9b}$$

$$a^{1/2}a^{\alpha\beta}\theta_{3,\beta} = [a^{1/2}a^{\alpha\beta}\mathbf{a}_{3,\beta}] \cdot \mathbf{g}^*, \tag{4.9c}$$

$$p^{3\alpha}_{,\alpha} = -\frac{k}{\theta_0}\left[\frac{H^3}{12}(a^{1/2}a^{\sigma\gamma}\mathbf{a}_{3,\sigma})_{,\gamma} + \frac{H^2}{4}H_{,\gamma}(a^{1/2}a^{\sigma\gamma}\mathbf{a}_{3,\sigma})\right] \cdot \mathbf{g}^*. \tag{4.9d}$$

Thus, with the help of (4.9b) it follows that (4.8a) is satisfied for all shells and all values of $\mathbf{g}^*$. Consequently, the constitutive equation for p^α need not be modified. Moreover, assuming that the constitutive equation for $p^{3\alpha}$ also needs no modification, Eq. (4.8b) requires

$$p^3 = -\frac{kH}{\theta_0}\left[\left\{a^{1/2} + \frac{H^2}{4}(\mathbf{a}_{3,1} \times \mathbf{a}_{3,2} \cdot \mathbf{a}_3)\right\}\mathbf{a}_3 + \frac{H^2}{12}(a^{1/2}a^{\sigma\gamma}\mathbf{a}_{3,\sigma})_{,\gamma} + \frac{H}{4}\{H_{,\gamma}(a^{1/2}a^{\sigma\gamma}\mathbf{a}_{3,\sigma}) - H_{,1}(\mathbf{a}_{3,2} \times \mathbf{a}_3) - H_{,2}(\mathbf{a}_3 \times \mathbf{a}_{3,1})\}\right] \cdot \mathbf{g}^*. \tag{4.10}$$

In particular, it can be seen that neither of the constitutive equations (3.13) or (3.14) satisfy the restriction (4.10) for general shell geometry.

To motivate a modified constitutive equation for p^3 it is noted that, for the general linear theory, p^3 is a linear function of $\{\theta_{,\alpha}, \theta_3, \theta_{3,\alpha}\}$ which can be expressed in the form

$$p^3 = -\frac{kH}{\theta_0}[B^\alpha\theta_{,\alpha} + B^3\theta_3 + B^{3\alpha}\theta_{3,\alpha}], \tag{4.11}$$

where $\{B^\alpha, B^3, B^{3\alpha}\}$ are functions of the shell's geometry. It therefore, follows from (4.5) that for constant temperature gradient p^3 takes the form:

$$p^3 = -\frac{kH}{\theta_0}[B^\alpha \mathbf{a}_\alpha + B^3\mathbf{a}_3 + B^{3\alpha}\mathbf{a}_{3,\alpha}] \cdot \mathbf{g}^*. \tag{4.12}$$

However, since $\mathbf{a}_{3,\alpha}$ are vectors tangent to the shells middle surface they can be expressed in terms of their components in the $\mathbf{a}_\alpha$ directions. This emphasizes that the expressions for $\{B^\alpha, B^3, B^{3\alpha}\}$ cannot be determined uniquely. Moreover, in proposing a constitutive equation for p^3 it is also necessary to recall that the complete set of constitutive equations must satisfy the second law of thermodynamics (3.11). Therefore, for simplicity (4.10) is solved by specifying

$$\begin{aligned} B^\alpha &= \left[\frac{H^2}{12}(a^{1/2}a^{\sigma\gamma}\mathbf{a}_{3,\sigma})_{,\gamma} + \frac{H}{4}\{H_{,\sigma}a^{1/2}a^{\sigma\gamma}\mathbf{a}_{3,\gamma} - H_{,1}(\mathbf{a}_{3,2}\times\mathbf{a}_3) - H_{,2}(\mathbf{a}_3\times\mathbf{a}_{3,1})\}\right]\cdot\mathbf{a}^\alpha, \\ B^3 &= \left[a^{1/2} + \frac{H^2}{4}(\mathbf{a}_{3,1}\times\mathbf{a}_{3,2}\cdot\mathbf{a}_3) + \frac{H^2}{12}\left\{(a^{1/2}a^{\sigma\gamma}\mathbf{a}_{3,\sigma})_{,\gamma}\cdot\mathbf{a}_3\right\}\right], \quad B^{3\alpha} = 0, \end{aligned} \tag{4.13}$$

which yield a non-trivial generalization of (3.13) and (3.14).

Next, it is necessary to check the validity of the second law of thermodynamics for the constitutive equations (3.12) and (4.11). Specifically, using (3.11) it follows that

$$-p^\alpha\theta_{,\alpha} - p^3\theta_3 - p^{3\alpha}\theta_{3,\alpha} = \frac{kH}{\theta_0}[a^{1/2}a^{\alpha\beta}]\theta_{,\alpha}\theta_{,\beta} + \frac{kH}{\theta_0}B^\alpha\theta_{,\alpha}\theta_3 + \frac{kH}{\theta_0}B^3(\theta_3)^2 + \frac{kH^3}{12\theta_0}[a^{1/2}a^{\alpha\beta}]\theta_{3,\alpha}\theta_{3,\beta}. \tag{4.14}$$

However, these terms can be rearranged to deduce that

$$\begin{aligned} -p^\alpha\theta_{,\alpha} - p^3\theta_3 - p^{3\alpha}\theta_{3,\alpha} &= \frac{kH}{\theta_0}\left[a^{1/4}\mathbf{a}^\alpha\theta_{,\alpha} + \frac{1}{2}a^{-1/4}\mathbf{B}\theta_3\right]\cdot\left[a^{1/4}\mathbf{a}^\beta\theta_{,\beta} + \frac{1}{2}a^{-1/4}\mathbf{B}\theta_3\right] \\ &\quad + \frac{kH}{\theta_0}G(\theta_3)^2 + \frac{kH^3}{12\theta_0}[a^{1/2}a^{\alpha\beta}]\theta_{3,\alpha}\theta_{3,\beta}, \end{aligned} \tag{4.15}$$

where the vector $\mathbf{B}$ and the scalar G are defined by

$$\mathbf{B} = B^\alpha\mathbf{a}_\alpha, \quad G = \left[B^3 - \frac{1}{4}a^{-1/2}\mathbf{B}\cdot\mathbf{B}\right]. \tag{4.16}$$

It then follows that (4.15) will remain non-negative (i.e. heat flows from hot to cold regions) provided that the function G is positive. This places limitations on the thickness H and on the variation of the thickness of the shell relative to its local radii of curvature. In the remainder of this work it is assumed that this condition is satisfied. Further in this regard, it is noted that the shell's thickness is also limited by the condition that the convected coordinates θ^i provide a one-to-one mapping to all material points, which requires $g^{*1/2}$ in (A.4) to remain positive.

In summary, the constitutive equations for the Cosserat model are given by (3.12) and (4.11) with the specifications (4.13), and the equations of heat conduction are given by (3.19). Moreover, for steady-state problems in the absence of external entropy supplies these equations can be solved for the heat flues $\bar{q}$ and $\hat{q}$ to obtain (3.20). Also, the temperature fields θ and θ_3 are given by (3.21).

5. Summary of two other models

The objective of this section is to summarize two other models for heat conduction in rigid shells proposed by Lukasiewicz (1989) and Hashin (2001). Lukasiewicz (1989) developed a general thermoelastic model for deformable shells as well a simplified model for transient heat conduction in rigid shells. Here, attention is confined to the steady-state form of the model for rigid shells with constant uniform normal thickness. By way of background, it is first noted that for orthogonal coordinates

$$\mathbf{a}_1 \cdot \mathbf{a}_2 = 0, \quad \left[\frac{\mathbf{a}_1}{\sqrt{a_{11}}}\right]_{,1} = -\frac{\sqrt{a_{11}}}{R_1}\mathbf{a}_3, \quad \left[\frac{\mathbf{a}_2}{\sqrt{a_{22}}}\right]_{,2} = -\frac{\sqrt{a_{22}}}{R_2}\mathbf{a}_3, \quad \mathbf{a}_{3,1} = \frac{1}{R_1}\mathbf{a}_1, \quad \mathbf{a}_{3,2} = \frac{1}{R_2}\mathbf{a}_2, \tag{5.1}$$

where the magnitudes of R_1 and R_2 are local radii of curvature. Now, the variables $\{h, H, T_1, T_2, q_n^-, q_n^+\}$ in Lukasiewicz (1989) are related to the variables used here by the expressions

$$h = H, \quad H = -\frac{1}{2}\left[\frac{1}{R_1} + \frac{1}{R_2}\right], \quad T_1 = \theta - \theta_0, \quad T_2 = \frac{H}{2}\theta_3, \quad q_n^- = -\bar{q}, \quad q_n^+ = \hat{q}. \tag{5.2}$$

Then, the equations (6.27) [The expression r_0 there should be h/k instead of k/h as stated; Lukasiewicz (2003, personal communication)] in Lukasiewicz (1989) can be rewritten in the forms

$$\begin{aligned} \bar{q} &= -\frac{k}{H}\left[-\frac{1}{2}\left(\frac{\kappa_1 + \kappa_2}{2}\right)^2(\theta - \theta_0) + \frac{H^2}{2}\nabla_s^2\theta + H\theta_3 - \frac{H^3}{10}\nabla_s^2\theta_3\right], \\ \hat{q} &= -\frac{k}{H}\left[\frac{1}{2}\left(\frac{\kappa_1 + \kappa_2}{2}\right)^2(\theta - \theta_0) - \frac{H^2}{2}\nabla_s^2\theta + H\theta_3 - \frac{H^3}{10}\nabla_s^2\theta_3\right], \end{aligned} \tag{5.3}$$

where the Laplacian ∇_s^2 associated with the shell's middle surface is defined by (3.15) and κ_1 and κ_2 are normalized measures of the thickness relative to the local radii of curvature

$$\kappa_1 = \frac{H}{R_1}, \quad \kappa_2 = \frac{H}{R_2}. \tag{5.4}$$

Hashin (2001) developed a model for a thin interphase which unified the more classical models associated with weakly conducting interfaces (Sanchez-Palencia, 1970) and highly conducting interfaces (Pham Huy and Sanchez-Palencia, 1974). The equations in Hashin (2001) were developed using an orthogonal curvilinear coordinate system. To compare this model with the Cosserat model it is noted that the variables $\{\alpha_1, \alpha_2, \alpha_3, k_i, t, \phi^1, \phi^2, q_n^1, q_n^2\}$ in Hashin (2001) can be related to the variables used here by the expressions

$$\alpha_1 = \theta^1, \quad \alpha_2 = \theta^2, \quad \alpha_3 = \theta^3, \quad k_i = k, \quad t = H, \quad \phi^1 = \bar{\theta}, \quad \phi^2 = \hat{\theta}, \quad q_n^1 = \bar{q}, \quad q_n^2 = \hat{q}. \tag{5.5}$$

Also, with the help of (5.1) the parameters $\{h_1, h_2\}$ in Hashin (2001) are given by

$$h_1 = \sqrt{a_{11}\left(1 + \frac{\theta^3}{R_1}\right)^2}, \quad h_2 = \sqrt{a_{22}\left(1 + \frac{\theta^3}{R_2}\right)^2}, \tag{5.6}$$

so that

$$\begin{aligned} \left.\frac{1}{h_1 h_2}\right|_{\theta^3=-H/2} &= \frac{1}{\sqrt{a_{11}a_{22}}}\left(\frac{2}{2-\kappa_1}\right)\left(\frac{2}{2-\kappa_2}\right), \\ \left.\frac{1}{h_1 h_2}\frac{\partial(h_1 h_2)}{\partial\theta^3}\right|_{\theta^3=-H/2} &= \frac{1}{H}\left[\frac{2\kappa_1}{2-\kappa_1} + \frac{2\kappa_2}{2-\kappa_2}\right], \end{aligned} \tag{5.7}$$

Next, using these results the steady-state forms of Eqs. (5) and (17) in Hashin (2001) can be rewritten as

$$\begin{aligned} \bar{q} &= -k\theta_3, \\ \hat{q} &= -k\left[1 - \frac{2\kappa_1}{2-\kappa_1} - \frac{2\kappa_2}{2-\kappa_2}\right]\theta_3 + kH\left(\frac{2}{2-\kappa_1}\right)\left(\frac{2}{2-\kappa_2}\right)\nabla_s^2\bar{\theta}, \end{aligned} \tag{5.8}$$

where the Laplacian associated with the shell's middle surface is defined in (3.15) and θ_3 is specified by (3.21). In particular, it is noted that these equations were developed using a Taylor series expansion about the surface $\theta^3 = -H/2$ so they are biased towards that surface.

6. Example of transient heat conduction in a plate

The coefficient for η^3 in (3.12) is different from the associated coefficient determined by Green and Naghdi (1979). Although this coefficient was used in Rubin (1986) it was determined in unpublished results which compared the Cosserat solution with an exact solution in Carslaw and Jaeger (1956, p. 112). Here, this value can be justified by considering a simpler problem of transient heat conduction in a plate of constant thickness H. Specifically, consider the problem of an insulated plate which initially has the temperature distribution specified by

$$\theta^*(\theta^3, 0) = \theta_0 + \frac{\beta}{2} \sin\left(\frac{\pi\theta^3}{H}\right), \tag{6.1}$$

where β controls the magnitude of the temperature field. Also, the boundary conditions require

$$\mathbf{q}^*\left(-\frac{H}{2}, t\right) = \bar{q}(t)\mathbf{e}_3 = 0, \quad \mathbf{q}^*\left(\frac{H}{2}, t\right) = \bar{q}(t)\mathbf{e}_3 = 0. \tag{6.2}$$

It is easy to see that the exact solution of (2.17) for this problem is given by

$$\theta^* = \theta_0 + A(t)\sin\left(\frac{\pi\theta^3}{H}\right), \quad A(t) = \frac{\beta}{2}\exp\left(-\frac{k\pi^2 t}{\rho^* c H^2}\right). \tag{6.3}$$

It therefore follows with the help of (3.21) that this exact solution is consistent with the results:

$$\bar{\theta}(\theta^\alpha, t) = \theta_0 - A(t), \quad \hat{\theta}(\theta^\alpha, t) = \theta_0 + A(t), \quad \theta(\theta^\alpha, t) = \theta_0, \quad \theta_3(\theta^\alpha, t) = \frac{2A(t)}{H}. \tag{6.4}$$

For the Cosserat model of a plate, the position vector $\mathbf{x}$ and the unit normal $\mathbf{a}_3$ are given in terms of the fixed rectangular Cartesian base vectors $\mathbf{e}_i$ by

$$\mathbf{x} = \theta^\alpha \mathbf{e}_\alpha, \quad \mathbf{a}_\alpha = \mathbf{e}_\alpha, \quad \mathbf{a}_3 = \mathbf{e}_3. \tag{6.5}$$

Thus, the balance laws (3.19) for a plate can be rewritten in the forms

$$\begin{aligned} &\rho^* H c\dot{\theta} = \theta_0 m s_s - (\hat{q} - \bar{q}) + kH\nabla_s^2\theta, \\ &\rho^* H \frac{cH^2}{\pi^2}\dot{\theta}_3 = \theta_0 m s_s^3 - \frac{H}{2}(\hat{q} + \bar{q}) - kH\theta_3 + \frac{kH^3}{12}\nabla_s^2\theta_3. \end{aligned} \tag{6.6}$$

Specifically, for the problem under consideration the external entropy supplies vanish and the temperature fields are $\{\theta, \theta_3\}$ are functions of time only so these equations reduce to

$$\dot{\theta} = 0, \quad \dot{\theta}_3 = -\frac{k\pi^2}{\rho^* c H^2}\theta_3. \tag{6.7}$$

Moreover, the initial conditions are specified to be consistent with the exact solution (6.4) with

$$\theta(0) = \theta_0, \quad \theta_3(0) = \frac{\beta}{H}, \tag{6.8}$$

so that the solution of (6.7) gives

$$\theta(t) = \theta_0, \quad \theta_3(t) = \frac{\beta}{H}\exp\left(-\frac{k\pi^2 t}{\rho^* c H^2}\right), \tag{6.9}$$

which is identical to the exact results (6.4).

7. Equations for a general cylindrical shell with constant thickness

The objective of this section is to reduce the theory of Sections 3 and 4 to the setting of a general cylindrical shell which is infinitely long in the axial direction but has constant normal thickness H. Specifically, the axial direction of the shell is taken to be the unit constant vector $\mathbf{e}_3$ with the axial coordinate z. Moreover, the coordinate θ^3 is retained as the normal coordinate to the shell's middle surface ($\theta^3 = 0$), and the two other coordinates defining points on this middle surface are specified by

$$\theta^1 = s, \quad \theta^2 = z, \tag{7.1}$$

where s is the arclength coordinate of the curve defined by the intersection of the middle surface with the $z = 0(\mathbf{e}_1 - \mathbf{e}_2)$ plane. Thus, material points on the shell's middle surface are located by $\mathbf{x}$ and the kinematic definitions (3.2) yield

$$\mathbf{x} = \mathbf{x}(s, z), \quad \mathbf{a}_1 = \frac{\partial \mathbf{x}}{\partial s}, \quad \mathbf{a}_2 = \frac{\partial \mathbf{x}}{\partial z} = \mathbf{e}_3, \quad \mathbf{a}_3 = \mathbf{a}_1 \times \mathbf{a}_2, \quad a^{1/2} = 1, \quad \mathbf{a}^\alpha = \mathbf{a}_\alpha, \tag{7.2}$$

where $\mathbf{a}_i$ are a set of right-handed orthonormal vectors. It then follows that

$$\frac{\partial \mathbf{a}_1}{\partial s} = -\frac{1}{R}\mathbf{a}_3, \quad \frac{\partial \mathbf{a}_1}{\partial z} = 0, \quad \frac{\partial \mathbf{a}_3}{\partial s} = \frac{1}{R}\mathbf{a}_1, \quad \frac{\partial \mathbf{a}_3}{\partial z} = 0, \quad R = R(s), \tag{7.3}$$

where R is positive for convex curves (relative to the origin) and negative for concave curves, and its magnitude is equal to the variable local radius of curvature of the shell's middle surface. Also, the temperature fields are given by

$$\theta = \theta(s, z, t), \quad \theta_3 = \theta_3(s, z, t). \tag{7.4}$$

Moreover, for this geometry it can be shown using (3.12), (3.18), (4.11) and (4.13) that

$$\begin{aligned}
&p^1 = -\left[\frac{kH}{\theta_0}\right]\frac{\partial \theta}{\partial s}, \quad p^2 = -\left[\frac{kH}{\theta_0}\right]\frac{\partial \theta}{\partial z}, \quad p^{31} = -\left[\frac{kH^3}{12\theta_0}\right]\frac{\partial \theta_3}{\partial s}, \quad p^{32} = -\left[\frac{kH^3}{12\theta_0}\right]\frac{\partial \theta_3}{\partial z}, \\
&p^3 = -\left[\frac{kH}{\theta_0}\right]\left[1 - \frac{\kappa^2}{12}\right]\theta_3 + \left[\frac{kH}{\theta_0}\right]\left[\frac{\kappa^2}{12}\frac{\partial R}{\partial s}\right]\frac{\partial \theta}{\partial s}, \\
&\bar{\alpha} = \frac{2-\kappa}{2}, \quad \hat{\alpha} = \frac{2+\kappa}{2},
\end{aligned} \tag{7.5}$$

where κ is the ratio of the thickness to the local radius of curvature

$$\kappa = \frac{H}{R}. \tag{7.6}$$

Thus, for cylindrical geometry the steady-state equation (3.20) reduce to

$$\begin{aligned}
\bar{q} &= -\frac{k}{H}\left(\frac{2}{2-\kappa}\right)\left[\frac{H^2}{2}\left\{\frac{\partial^2 \theta}{\partial s^2} + \frac{\partial^2 \theta}{\partial z^2}\right\} + \left\{\left(1 - \frac{\kappa^2}{12}\right)H\theta_3 - \frac{\kappa^2 H}{12}\frac{\partial R}{\partial s}\frac{\partial \theta}{\partial s}\right\} - \frac{H^3}{12}\left\{\frac{\partial^2 \theta_3}{\partial s^2} + \frac{\partial^2 \theta_3}{\partial z^2}\right\}\right], \\
\hat{q} &= -\frac{k}{H}\left(\frac{2}{2+\kappa}\right)\left[-\frac{H^2}{2}\left\{\frac{\partial^2 \theta}{\partial s^2} + \frac{\partial^2 \theta}{\partial z^2}\right\} + \left\{\left(1 - \frac{\kappa^2}{12}\right)H\theta_3 - \frac{\kappa^2 H}{12}\frac{\partial R}{\partial s}\frac{\partial \theta}{\partial s}\right\} - \frac{H^3}{12}\left\{\frac{\partial^2 \theta_3}{\partial s^2} + \frac{\partial^2 \theta_3}{\partial z^2}\right\}\right].
\end{aligned} \tag{7.7}$$

Also, the function G in (4.16) is given by

$$G = 1 - \frac{\kappa^2}{12} - \frac{\kappa^4}{576}\left(\frac{\mathrm{d}R}{\mathrm{d}s}\right)^2, \tag{7.8}$$

which must remain positive.

Similarly, for this geometry $\kappa_1 = \kappa$ and $\kappa_2 = 0$ so that Lukasiewicz's (1989) equations (5.3) become

$$\begin{aligned}\bar{q} &= -\frac{k}{H}\left[-\frac{\kappa^2}{8}(\theta-\theta_0)+\frac{H^2}{2}\left(\frac{\partial^2\theta}{\partial s^2}+\frac{\partial^2\theta}{\partial z^2}\right)+H\theta_3-\frac{H^3}{10}\left(\frac{\partial^2\theta_3}{\partial s^2}+\frac{\partial^2\theta_3}{\partial z^2}\right)\right],\\ \hat{q} &= -\frac{k}{H}\left[\frac{\kappa^2}{8}(\theta-\theta_0)-\frac{H^2}{2}\left(\frac{\partial^2\theta}{\partial s^2}+\frac{\partial^2\theta}{\partial z^2}\right)+H\theta_3-\frac{H^3}{10}\left(\frac{\partial^2\theta_3}{\partial s^2}+\frac{\partial^2\theta_3}{\partial z^2}\right)\right]\end{aligned} \tag{7.9}$$

and Hashin's (2001) equations (5.8) become

$$\bar{q} = -\frac{k}{H}[H\theta_3], \quad \hat{q} = -\frac{k}{H}\left(\frac{2}{2-\kappa}\right)\left[\left(\frac{2-3\kappa}{2}\right)H\theta_3 - H^2\left(\frac{\partial^2\bar{\theta}}{\partial s^2}+\frac{\partial^2\bar{\theta}}{\partial z^2}\right)\right]. \tag{7.10}$$

8. Examples of a circular cylindrical shell

Consider the problem of a circular cylindrical shell which has inner radius a and outer radius b. Then, the constant mean radius R and uniform thickness H satisfy the equations

$$R = \frac{a+b}{2}, \quad H = b-a, \quad \kappa = \frac{H}{R}, \quad a = R\left(\frac{2-\kappa}{2}\right), \quad b = R\left(\frac{2+\kappa}{2}\right). \tag{8.1}$$

Now, for cylindrical polar coordinates the three-dimensional position vector $\mathbf{x}^*$ takes the form

$$\mathbf{x}^* = r\mathbf{e}_r(\phi) + z\mathbf{e}_3, \tag{8.2}$$

where the unit base vectors $\mathbf{e}_r$ and $\mathbf{e}_\phi$ are given by

$$\mathbf{e}_r(\phi) = \cos\phi\mathbf{e}_1 + \sin\phi\mathbf{e}_2, \quad \mathbf{e}_\phi(\phi) = -\sin\phi\mathbf{e}_1 + \cos\phi\mathbf{e}_2. \tag{8.3}$$

Also, the polar angle is denoted by ϕ in order to avoid confusion with the temperature field θ.

In this section attention will be confined to two-dimensional steady-state problems for which the temperature fields are independent of z and t. For this class of problems the exact temperature field θ^* satisfies (2.17) which reduces to

$$\frac{\partial^2\theta^*}{\partial r^2} + \frac{1}{r}\frac{\partial\theta^*}{\partial r} + \frac{1}{r^2}\frac{\partial^2\theta^*}{\partial\phi^2} = 0. \tag{8.4}$$

Moreover, taking

$$\theta^1 = \phi, \quad \theta^2 = z, \quad s = R\phi. \tag{8.5}$$

The Cosserat equations (7.7) require

$$\begin{aligned}\bar{q} &= -\frac{k}{H}\left(\frac{2}{2-\kappa}\right)\left[\frac{\kappa^2}{2}\frac{\partial^2\theta}{\partial\phi^2}+\left(1-\frac{\kappa^2}{12}\right)H\theta_3-\frac{\kappa^2 H}{12}\frac{\partial^2\theta_3}{\partial\phi^2}\right],\\ \hat{q} &= -\frac{k}{H}\left(\frac{2}{2+\kappa}\right)\left[-\frac{\kappa^2}{2}\frac{\partial^2\theta}{\partial\phi^2}+\left(1-\frac{\kappa^2}{12}\right)H\theta_3-\frac{\kappa^2 H}{12}\frac{\partial^2\theta_3}{\partial\phi^2}\right],\end{aligned} \tag{8.6}$$

Lukasiewicz's (1989) equations (7.9) require

$$\bar{q} = -\frac{k}{H}\left[-\frac{\kappa^2}{8}(\theta - \theta_0) + \frac{\kappa^2}{2}\frac{\partial^2\theta}{\partial\phi^2} + H\theta_3 - \frac{\kappa^2 H}{10}\frac{\partial^2\theta_3}{\partial\phi^2}\right],$$
$$\hat{q} = -\frac{k}{H}\left[\frac{\kappa^2}{8}(\theta - \theta_0) - \frac{\kappa^2}{2}\frac{\partial^2\theta}{\partial\phi^2} + H\theta_3 - \frac{\kappa^2 H}{10}\frac{\partial^2\theta_3}{\partial\phi^2}\right] \tag{8.7}$$

and Hashin's (2001) equations (7.10) require

$$\bar{q} = -\frac{k}{H}[H\theta_3], \quad \hat{q} = -\frac{k}{H}\left(\frac{2}{2-\kappa}\right)\left[\left(\frac{2-3\kappa}{2}\right)H\theta_3 - \kappa^2\frac{\partial^2\bar{\theta}}{\partial\phi^2}\right]. \tag{8.8}$$

Also, the function G in (7.8) remains positive since $\mathrm{d}R/\mathrm{d}s = 0$ and $\kappa \leqslant 2$.

In Section 3 the Cosserat equations were modified to ensure that they reproduce the exact solution for a constant temperature gradient in any direction. Therefore, to explore the differences between these three models it is possible to consider the simple steady-state solution of constant temperature gradient β in the $\mathbf{e}_1$ direction for which the temperature fields are given by

$$\theta^* = \theta_0 + \beta r\cos\phi, \quad \theta = \theta_0 + \beta R\cos\phi, \quad \theta_3 = \beta\cos\phi, \quad \bar{\theta} = \theta_0 + \beta R\left(\frac{2-\kappa}{2}\right)\cos\phi,$$
$$\hat{\theta} = \theta_0 + \beta R\left(\frac{2+\kappa}{2}\right)\cos\phi \tag{8.9}$$

and the exact solutions for the heat fluxes become

$$\bar{q} = \bar{Q}\cos\phi, \quad \hat{q} = \hat{Q}\cos\phi, \quad \bar{Q} = -k\beta, \quad \hat{Q} = -k\beta. \tag{8.10a,b,c,d}$$

Next, substituting (8.9) into the Cosserat equation (8.6) yields the exact solution (8.10) for $\bar{q}$ and $\hat{q}$. In contrast, Lukasiewicz's (1989) equations (8.7) predict the forms (8.10a,b) with

$$\bar{Q} = -\left[1 - \frac{5\kappa}{8} + \frac{\kappa^2}{10}\right]k\beta, \quad \hat{Q} = -\left[1 + \frac{5\kappa}{8} + \frac{\kappa^2}{10}\right]k\beta \tag{8.11a,b}$$

and Hashin's (2001) equations (8.8) predict the forms (8.10a,b) with

$$\bar{Q} = -k\beta, \quad \hat{Q} = -\left[\frac{2 - 5\kappa + \kappa^2}{2-\kappa}\right]k\beta, \tag{8.12a,b}$$

neither of which predict the exact solution (8.10). Moreover, a series expansion of these solutions indicates the solutions (8.11a,b) and (8.12b) are only accurate to zero-order in κ. This means that the solutions (8.11a,b) and (8.12b) do not predict the correct slope in the thin shell limit ($\kappa \to 0$).

The equations for rigid heat conducting shells should produce reasonably accurate predictions when the shell is thin. However, it is well known that the notion of ''thin'' is not purely geometrical and that the shell theory is accurate when the variation in quantities through the shell's thickness is not too severe. Therefore, it is of interest to explore the limits of shell theory by considering a class of problems which allows for control of the variation of the temperature field through the shell's thickness. To this end, consider the class of exact solutions of (8.4) for which

$$\theta^* = \theta_0 + \Theta^*(r)\cos(n\phi), \quad \Theta^*(r) = \left[A\left(\frac{r}{R}\right)^n + B\left(\frac{R}{r}\right)^n\right], \quad q^* = Q^*(r)\cos(n\phi),$$
$$Q^*(r) = -\frac{k}{H}\kappa n\left[A\left(\frac{r}{R}\right)^{n-1} - B\left(\frac{R}{r}\right)^{n+1}\right], \tag{8.13}$$

where the constants A and B are determined by boundary conditions, q^* represents the radial component of the heat flux vector and the integer n controls the variation of the temperature field through the shell's thickness. For the problem under consideration the boundary conditions are specified

$$\Theta^*(a) = 0, \quad \Theta^*(b) = \beta, \tag{8.14}$$

where β controls the magnitude of the temperature at the outer surface. It then follows that the exact solution of this problem is obtained by solving the equations

$$A\left(\frac{2-\kappa}{2}\right)^n + B\left(\frac{2}{2-\kappa}\right)^n = 0, \quad A\left(\frac{2+\kappa}{2}\right)^n + B\left(\frac{2}{2+\kappa}\right)^n = \beta \tag{8.15}$$

to obtain

$$A = \left[\frac{\left(\frac{2}{2-\kappa}\right)^n}{\left(\frac{2+\kappa}{2-\kappa}\right)^n - \left(\frac{2-\kappa}{2+\kappa}\right)^n}\right]\beta, \quad B = -\left[\frac{\left(\frac{2-\kappa}{2}\right)^n}{\left(\frac{2+\kappa}{2-\kappa}\right)^n - \left(\frac{2-\kappa}{2+\kappa}\right)^n}\right]\beta. \tag{8.16}$$

Consequently, the normal components $\bar{q}$ and $\hat{q}$ of the heat fluxes at the inner and outer boundaries, respectively, take the forms

$$\bar{q} = \overline{Q}\cos(n\phi), \quad \hat{q} = \widehat{Q}\cos(n\phi), \tag{8.17}$$

where the constants $\overline{Q}$ and $\widehat{Q}$ are given by

$$\begin{aligned} \overline{Q} &= -\frac{k}{H}\kappa n\left[\frac{\frac{4}{2-\kappa}}{\left(\frac{2+\kappa}{2-\kappa}\right)^n - \left(\frac{2-\kappa}{2+\kappa}\right)^n}\right]\beta, \\ \widehat{Q} &= -\frac{k}{H}\kappa n\left(\frac{2}{2+\kappa}\right)\left[\frac{\left(\frac{2+\kappa}{2-\kappa}\right)^n + \left(\frac{2-\kappa}{2+\kappa}\right)^n}{\left(\frac{2+\kappa}{2-\kappa}\right)^n - \left(\frac{2-\kappa}{2+\kappa}\right)^n}\right]\beta. \end{aligned} \tag{8.18}$$

For the Cosserat shell model the temperature fields in (3.21) associated with (8.13) and (8.14) are specified by

$$\theta = \theta_0 + \frac{\beta}{2}\cos(n\phi), \quad \theta_3 = \frac{\beta}{H}\cos(n\phi), \quad \bar{\theta} = \theta_0, \quad \hat{\theta} = \theta_0 + \beta\cos(n\phi) \tag{8.19}$$

so that Eqs. (8.6) yield heat fluxes in the forms (8.17) with

$$\overline{Q} = -\frac{k}{H}\left(\frac{2}{2-\kappa}\right)\left[1 - \frac{\kappa^2}{12}(1+2n^2)\right]\beta, \quad \widehat{Q} = -\frac{k}{H}\left(\frac{2}{2+\kappa}\right)\left[1 - \frac{\kappa^2}{12}(1-4n^2)\right]\beta. \tag{8.20a,b}$$

Similarly, Lukasiewicz's (1989) equations (8.7) yield heat fluxes in the forms (8.17) with

$$\overline{Q} = -\frac{k}{H}\left[1 - \kappa^2\left\{\frac{1}{16} + \frac{3}{20}n^2\right\}\right]\beta, \quad \widehat{Q} = -\frac{k}{H}\left[1 + \kappa^2\left\{\frac{1}{16} + \frac{7}{20}n^2\right\}\right]\beta \tag{8.21a,b}$$

and Hashin's (2001) equations (8.8) yield heat fluxes in the forms (8.17) with

$$\overline{Q} = -\frac{k}{H}\beta, \quad \widehat{Q} = -\frac{k}{H}\left[\frac{2-3\kappa}{2-\kappa}\right]\beta. \tag{8.22a,b}$$

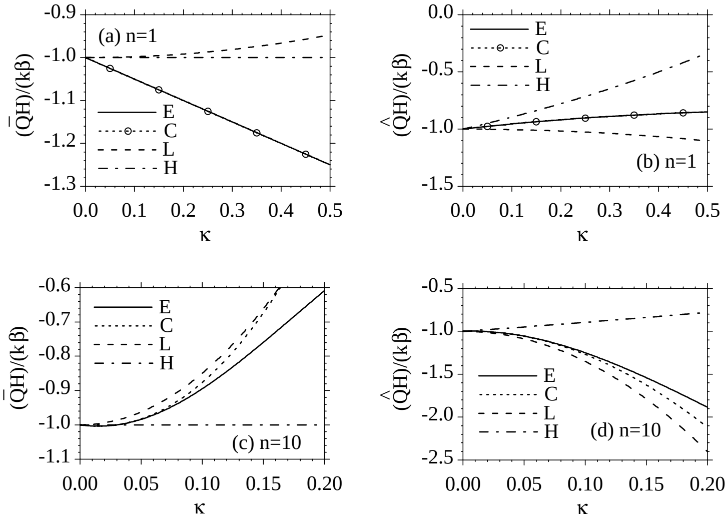

Fig. 1. Response of a circular cylindrical shell. Comparison of the exact (E), Cosserat (C), Lukasiewicz (L) and Hashin (H) solutions.

It can be shown that the Cosserat solution (8.20) predicts the exact solution (8.18) for all values of κ (all thicknesses of the shell) when $n = 1$. However, for other values of n the Cosserat solution is only approximate. A series expansion of these solutions indicates that the Cosserat solutions (8.20a,b) are both accurate to third-order in κ, whereas the other solutions (8.21a,b) and (8.22a,b) are only accurate to zero-order in κ. This means that the solutions (8.21) and (8.22) do not predict the correct slope in the thin shell limit ($\kappa \to 0$). Fig. 1 compares the predictions of these solutions for a range of shell thicknesses κ and for two values of n. For all cases the Cosserat solution converges smoothly to the exact solution as the shell becomes thin. Moreover, it is seen that the Cosserat theory predicts accurate results for moderately thick shells and moderately strong variation of the temperature field through the shell's thickness.

9. Equations for a spherical shell with constant thickness

For a spherical shell of mean radius R and constant thickness H, the normalized thickness κ is defined by

$$\kappa = \frac{H}{R} \tag{9.1}$$

and a material point on the spherical middle surface of the shell is located by the position vector $\mathbf{x}$

$$\mathbf{x} = R\mathbf{e}_r(\gamma, \phi), \tag{9.2}$$

where the orthonormal base vectors $\{\mathbf{e}_r, \mathbf{e}_\gamma, \mathbf{e}_\phi\}$ associated with the spherical polar coordinates $\{r, \gamma, \phi\}$ are defined in terms of the fixed base vectors $\mathbf{e}_i$ of a rectangular Cartesian coordinate system by the equations

$$\begin{aligned}
\mathbf{e}_r &= \mathbf{e}_r(\gamma, \phi) = \sin\gamma(\cos\phi\mathbf{e}_1 + \sin\phi\mathbf{e}_2) + \cos\gamma\mathbf{e}_3, \\
\mathbf{e}_\gamma &= \mathbf{e}_\gamma(\gamma, \phi) = \cos\gamma(\cos\phi\mathbf{e}_1 + \sin\phi\mathbf{e}_2) - \sin\gamma\mathbf{e}_3, \\
\mathbf{e}_\phi &= \mathbf{e}_\phi(\phi) = -\sin\phi\mathbf{e}_1 + \cos\phi\mathbf{e}_2.
\end{aligned} \tag{9.3}$$

Again, the symbol ϕ is used instead of the usual symbol θ for the circumferential coordinate to avoid confusion with the temperature fields introduced earlier. Also, it can be shown that

$$\frac{\partial \mathbf{e}_r}{\partial \gamma} = \mathbf{e}_\gamma, \quad \frac{\partial \mathbf{e}_r}{\partial \phi} = \sin\gamma \mathbf{e}_\phi, \quad \frac{\partial \mathbf{e}_\gamma}{\partial \gamma} = -\mathbf{e}_r, \quad \frac{\partial \mathbf{e}_\gamma}{\partial \phi} = \cos\gamma \mathbf{e}_\phi, \quad \frac{\partial \mathbf{e}_\phi}{\partial \gamma} = 0, \quad \frac{\partial \mathbf{e}_\phi}{\partial \phi} = -\sin\gamma \mathbf{e}_r - \cos\gamma \mathbf{e}_\gamma. \tag{9.4}$$

Now, using the definitions in Section 3, the geometry of the spherical shell is given by

$$\begin{aligned}
&\theta^1 = \gamma, \quad \theta^2 = \phi, \quad \mathbf{a}_1 = R\mathbf{e}_\gamma, \quad \mathbf{a}_2 = R\sin\gamma \mathbf{e}_\phi, \quad \mathbf{a}_3 = \mathbf{e}_r, \quad a^{1/2} = R^2 \sin\gamma, \\
&\mathbf{a}^1 = \frac{1}{R}\mathbf{e}_\gamma, \quad \mathbf{a}^2 = \frac{1}{R\sin\gamma}\mathbf{e}_\phi, \quad \mathbf{a}_{3,1} = \mathbf{e}_\gamma, \quad \mathbf{a}_{3,2} = \sin\gamma \mathbf{e}_\phi, \\
&\mathbf{a}^\sigma \cdot \mathbf{a}_{3,\sigma} = \frac{2}{R}, \quad \mathbf{a}_{3,1} \times \mathbf{a}_{3,2} \cdot \mathbf{a}_3 = \sin\gamma, \quad (a^{1/2} a^{\sigma\gamma} \mathbf{a}_{3,\sigma})_{,\gamma} = -2\sin\gamma \mathbf{e}_r, \\
&\bar{\alpha} = R^2 \left(\frac{2-\kappa}{2}\right)^2 \sin\gamma, \quad \hat{\alpha} = R^2 \left(\frac{2+\kappa}{2}\right)^2 \sin\gamma.
\end{aligned} \tag{9.5}$$

Also, the function G in (4.16) is given by

$$G = R^2 \sin\gamma \left[1 + \frac{\kappa^2}{12}\right], \tag{9.6}$$

which remains positive except for the singular points when $\sin\gamma = 0$. Consequently, the second law of thermodynamics is satisfied by these constitutive equations for all spherical shells. Moreover, the constitutive equations (3.12) for p^α and $p^{3\alpha}$ and (4.11) for p^3 yield

$$\begin{aligned}
p^\alpha_{,\alpha} &= -\frac{kH}{\theta_0} \sin\gamma \left[\frac{\partial^2 \theta}{\partial \gamma^2} + \cot\gamma \frac{\partial \theta}{\partial \gamma} + \frac{1}{\sin^2\gamma} \frac{\partial^2 \theta}{\partial \phi^2}\right], \\
p^{3\alpha}_{,\alpha} &= -\frac{kH^3}{12\theta_0} \sin\gamma \left[\frac{\partial^2 \theta_3}{\partial \gamma^2} + \cot\gamma \frac{\partial \theta_3}{\partial \gamma} + \frac{1}{\sin^2\gamma} \frac{\partial^2 \theta_3}{\partial \phi^2}\right], \\
p^3 &= -\left[\frac{kH}{\theta_0} R^2 \left(1 + \frac{\kappa^2}{12}\right) \sin\gamma\right] \theta_3,
\end{aligned} \tag{9.7}$$

so that the heat fluxes $\bar{q}$ and $\hat{q}$ in (3.20) can be expressed in the forms

$$\begin{aligned}
\bar{q} = -\frac{k}{H}\left(\frac{2}{2-\kappa}\right)^2 \Bigg[&\left(1 + \frac{\kappa^2}{12}\right) H\theta_3 + \frac{\kappa^2}{2}\left\{\frac{\partial^2 \theta}{\partial \gamma^2} + \cot\gamma \frac{\partial \theta}{\partial \gamma} + \frac{1}{\sin^2\gamma} \frac{\partial^2 \theta}{\partial \phi^2}\right\} \\
&- \frac{\kappa^2 H}{12}\left\{\frac{\partial^2 \theta_3}{\partial \gamma^2} + \cot\gamma \frac{\partial \theta_3}{\partial \gamma} + \frac{1}{\sin^2\gamma} \frac{\partial^2 \theta_3}{\partial \phi^2}\right\}\Bigg], \\
\hat{q} = -\frac{k}{H}\left(\frac{2}{2+\kappa}\right)^2 \Bigg[&\left(1 + \frac{\kappa^2}{12}\right) H\theta_3 - \frac{\kappa^2}{2}\left\{\frac{\partial^2 \theta}{\partial \gamma^2} + \cot\gamma \frac{\partial \theta}{\partial \gamma} + \frac{1}{\sin^2\gamma} \frac{\partial^2 \theta}{\partial \phi^2}\right\} \\
&- \frac{\kappa^2 H}{12}\left\{\frac{\partial^2 \theta_3}{\partial \gamma^2} + \cot\gamma \frac{\partial \theta_3}{\partial \gamma} + \frac{1}{\sin^2\gamma} \frac{\partial^2 \theta_3}{\partial \phi^2}\right\}\Bigg].
\end{aligned} \tag{9.8}$$

In these equations the temperature fields $\{\theta, \theta_3\}$ are defined by (3.21) in terms of the temperatures $\{\bar{\theta}, \hat{\theta}\}$ at the bottom and top surfaces of the shell, respectively.

For this spherical shell $\kappa_1 = \kappa_2 = \kappa$, Lukasiewicz's (1989) equations (5.3) become

$$\begin{aligned}\bar{q} &= -\frac{k}{H}\left[-\frac{\kappa^2}{2}(\theta-\theta_0) + \frac{\kappa^2}{2}\left\{\frac{\partial^2\theta}{\partial\gamma^2} + \cot\gamma\frac{\partial\theta}{\partial\gamma} + \frac{1}{\sin^2\gamma}\frac{\partial^2\theta}{\partial\phi^2}\right\} + H\theta_3\right.\\ &\left.\quad - \frac{\kappa^2 H}{10}\left\{\frac{\partial^2\theta_3}{\partial\gamma^2} + \cot\gamma\frac{\partial\theta_3}{\partial\gamma} + \frac{1}{\sin^2\gamma}\frac{\partial^2\theta_3}{\partial\phi^2}\right\}\right],\\ \hat{q} &= -\frac{k}{H}\left[\frac{\kappa^2}{2}(\theta-\theta_0) - \frac{\kappa^2}{2}\left\{\frac{\partial^2\theta}{\partial\gamma^2} + \cot\gamma\frac{\partial\theta}{\partial\gamma} + \frac{1}{\sin^2\gamma}\frac{\partial^2\theta}{\partial\phi^2}\right\} + H\theta_3\right.\\ &\left.\quad - \frac{\kappa^2 H}{10}\left\{\frac{\partial^2\theta_3}{\partial\gamma^2} + \cot\gamma\frac{\partial\theta_3}{\partial\gamma} + \frac{1}{\sin^2\gamma}\frac{\partial^2\theta_3}{\partial\phi^2}\right\}\right]\end{aligned} \tag{9.9}$$

and Hashin's (2001) equations (5.8) become

$$\begin{aligned}\bar{q} &= -\frac{k}{H}(\hat{\theta}-\bar{\theta}),\\ \hat{q} &= -\frac{k}{H}\left[\left(\frac{2-5\kappa}{2-\kappa}\right)(\hat{\theta}-\bar{\theta}) - \left(\frac{2\kappa}{2-\kappa}\right)^2\left\{\frac{\partial^2\bar{\theta}}{\partial\gamma^2} + \cot\gamma\frac{\partial\bar{\theta}}{\partial\gamma} + \frac{1}{\sin^2\gamma}\frac{\partial^2\bar{\theta}}{\partial\phi^2}\right\}\right],\end{aligned} \tag{9.10}$$

where the temperature fields are given by (3.21).

In Section 3 the Cosserat equations were modified to ensure that they reproduce the exact solution for a constant heat flux in any direction for arbitrary shell geometry. Therefore, to explore the differences between these three models it is possible to consider the simple steady-state solution of constant heat flux β for which the exact temperature field θ^* is specified by

$$\theta^* = \theta_0 + \beta x_3 = \theta_0 + \beta r\cos\gamma. \tag{9.11}$$

It then follows from (3.21) and (9.11) that

$$\begin{aligned}&\bar{\theta} = \theta_0 + \frac{H\beta}{\kappa}\left(\frac{2-\kappa}{2}\right)\cos\gamma, \quad \hat{\theta} = \theta_0 + \frac{H\beta}{\kappa}\left(\frac{2+\kappa}{2}\right)\cos\gamma,\\ &\theta = \theta_0 + \frac{H\beta}{\kappa}\cos\gamma, \quad H\theta_3 = H\beta\cos\gamma.\end{aligned} \tag{9.12}$$

Next, substituting these expressions into the Cosserat equations (9.8) yield

$$\bar{q} = \hat{q} = -k\beta\cos\gamma, \tag{9.13}$$

which is consistent with the exact solution. In contrast, Lukasiewicz's (1989) equations (9.9) yield

$$\bar{q} = -\left[1 - \frac{3\kappa}{2} + \frac{\kappa^2}{5}\right]k\beta\cos\gamma, \quad \hat{q} = -\left[1 + \frac{3\kappa}{2} + \frac{\kappa^2}{5}\right]k\beta\cos\gamma \tag{9.14.a,b}$$

and Hashin's (2001) equations (9.10) yield

$$\bar{q} = -k\beta\cos\gamma, \quad \hat{q} = -k\left[\frac{2-3\kappa}{2-\kappa}\right]\beta\cos\gamma, \tag{9.15a,b}$$

neither of which predict the exact solution (9.13). Moreover, a series expansion of these solutions indicates that the solutions (9.14a,b) or (9.15b) are only accurate to zero-order in κ. This means that the solutions (9.14) and (9.15) do not predict the correct slope in the thin shell limit ($\kappa \to 0$).

10. Example of a spherical shell

As a specific example consider the problem of a spherical shell which has inner radius a and outer radius b. Then, the mean radius R and uniform thickness H satisfy the equations

$$R = \frac{a+b}{2}, \quad H = b - a, \quad \kappa = \frac{H}{R}, \quad a = R\left(\frac{2-\kappa}{2}\right), \quad b = R\left(\frac{2+\kappa}{2}\right). \tag{10.1}$$

Now, for steady-state the exact temperature field θ^* satisfies the equation

$$\frac{\partial^2\theta^*}{\partial r^2} + \frac{2}{r}\frac{\partial\theta^*}{\partial r} + \frac{1}{r^2}\frac{\partial^2\theta^*}{\partial\gamma^2} + \frac{\cot\gamma}{r^2}\frac{\partial\theta^*}{\partial\gamma} + \frac{1}{r^2\sin^2\gamma}\frac{\partial^2\theta^*}{\partial\phi^2} = 0. \tag{10.2}$$

Therefore, following Carslaw and Jaeger (1959, p. 248) it can be shown using the change of variables

$$\mu = \cos\gamma, \tag{10.3}$$

that

$$\frac{\partial^2\theta^*}{\partial\gamma^2} + \cot\gamma\frac{\partial\theta^*}{\partial\gamma} = \frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial\theta^*}{\partial\mu}\right]. \tag{10.4}$$

Moreover, recalling that the Legendre functions $P_n(\mu)$ satisfy the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}\mu}\left[(1-\mu^2)\frac{\mathrm{d}P_n}{\mathrm{d}\mu}\right] = -n(n+1)P_n \tag{10.5}$$

for positive integer values of n it follows that a class of exact solutions of (10.2) can be written in the forms

$$\begin{aligned}
&\theta^* = \theta_0 + \Theta^*(r)P_n(\mu), \quad \Theta^*(r) = \left[A\left(\frac{r}{R}\right)^{m_1} + B\left(\frac{R}{r}\right)^{m_2}\right],\\
&q^* = Q^*(r)P_n(\mu), \quad Q^*(r) = -\frac{k}{H}\kappa\left[m_1A\left(\frac{r}{R}\right)^{m_1-1} - m_2B\left(\frac{R}{r}\right)^{m_2+1}\right],\\
&m_1 = \frac{-1+\sqrt{1+4n(n+1)}}{2}, \quad m_2 = \frac{1+\sqrt{1+4n(n+1)}}{2},
\end{aligned} \tag{10.6}$$

where the constants A and B are determined by boundary conditions. Also, q^* represents the radial component of the heat flux vector. For the problem under consideration the boundary conditions are specified

$$\Theta^*(a) = 0, \quad \Theta^*(b) = \beta, \tag{10.7}$$

where β controls the magnitude of the temperature at the outer surface. It then follows that the exact solution of this problem is obtained by solving the equations

$$A\left(\frac{2-\kappa}{2}\right)^{m_1} + B\left(\frac{2}{2-\kappa}\right)^{m_2} = 0, \quad A\left(\frac{2+\kappa}{2}\right)^{m_1} + B\left(\frac{2}{2+\kappa}\right)^{m_2} = \beta \tag{10.8}$$

to obtain

$$A = \left[\frac{\left(\frac{2}{2-\kappa}\right)^{m_1}}{\left(\frac{2+\kappa}{2-\kappa}\right)^{m_1} - \left(\frac{2-\kappa}{2+\kappa}\right)^{m_2}} \right] \beta, \quad B = -\left[\frac{\left(\frac{2-\kappa}{2}\right)^{m_2}}{\left(\frac{2+\kappa}{2-\kappa}\right)^{m_1} - \left(\frac{2-\kappa}{2+\kappa}\right)^{m_2}} \right] \beta. \tag{10.9}$$

Consequently, the normal components $\bar{q}$ and $\hat{q}$ of the heat fluxes at the inner and outer boundaries, respectively, take the forms

$$\bar{q} = \overline{Q} P_n(\mu), \quad \hat{q} = \widehat{Q} P_n(\mu), \tag{10.10}$$

where the constants $\overline{Q}$ and $\widehat{Q}$ are given by

$$\begin{aligned} \overline{Q} &= -\frac{k}{H}\kappa \left[m_1 A \left(\frac{2-\kappa}{2} \right)^{m_1-1} - m_2 B \left(\frac{2}{2-\kappa} \right)^{m_2+1} \right], \\ \widehat{Q} &= -\frac{k}{H}\kappa \left[m_1 A \left(\frac{2+\kappa}{2} \right)^{m_1-1} - m_2 B \left(\frac{2}{2+\kappa} \right)^{m_2+1} \right]. \end{aligned} \tag{10.11}$$

For the Cosserat shell model the temperature fields in (3.21) are specified by

$$\theta = \theta_0 + \frac{\beta}{2} P_n(\mu), \quad \theta_3 = \frac{\beta}{H} P_n(\mu), \quad \bar{\theta} = \theta_0, \quad \hat{\theta} = \theta_0 + \beta P_n(\mu), \tag{10.12}$$

so that Eqs. (9.8) yield heat fluxes in the forms (10.10) with

$$\overline{Q} = -\frac{k}{H} \left(\frac{2}{2-\kappa} \right)^2 \left[1 + \frac{\kappa^2}{12} \{ 1 - 2n(n+1) \} \right] \beta, \tag{10.13a}$$

$$\widehat{Q} = -\frac{k}{H} \left(\frac{2}{2+\kappa} \right)^2 \left[1 + \frac{\kappa^2}{12} \{ 1 + 4n(n+1) \} \right] \beta. \tag{10.13b}$$

Similarly, Lukasiewicz's (1989) equations (9.9) yield heat fluxes in the forms (10.10) with

$$\overline{Q} = -\frac{k}{H} \left[1 - \frac{\kappa^2}{20} \{ 5 + 3n(n+1) \} \right] \beta, \tag{10.14a}$$

$$\widehat{Q} = -\frac{k}{H} \left[1 + \frac{\kappa^2}{20} \{ 5 + 7n(n+1) \} \right] \beta, \tag{10.14b}$$

and Hashin's (2001) equations (9.10) yield heat fluxes in the forms (10.10) withs

$$\overline{Q} = -\frac{k}{H} \beta, \quad \widehat{Q} = -\frac{k}{H} \left[\frac{2-5\kappa}{2-\kappa} \right] \beta, \tag{10.15a,b}$$

A series expansion of these solutions indicates that the Cosserat solutions (10.13a,b) are both accurate to first-order in κ for any value of n and they are accurate to second-order in κ for large values of n. In contrast, the other solutions (10.14a,b) and (10.15a,b) are only accurate to zero-order in κ. This means that the solutions (10.14) and (10.15) do not predict the correct slope in the thin shell limit ($\kappa \to 0$). Fig. 2 compares the predictions of these solutions for a range of shell thicknesses κ and for two values of n. For all cases the Cosserat solution converges smoothly to the exact solution as the shell becomes thin. Again, it is seen that the Cosserat theory predicts accurate results for moderately thick shells and moderately strong

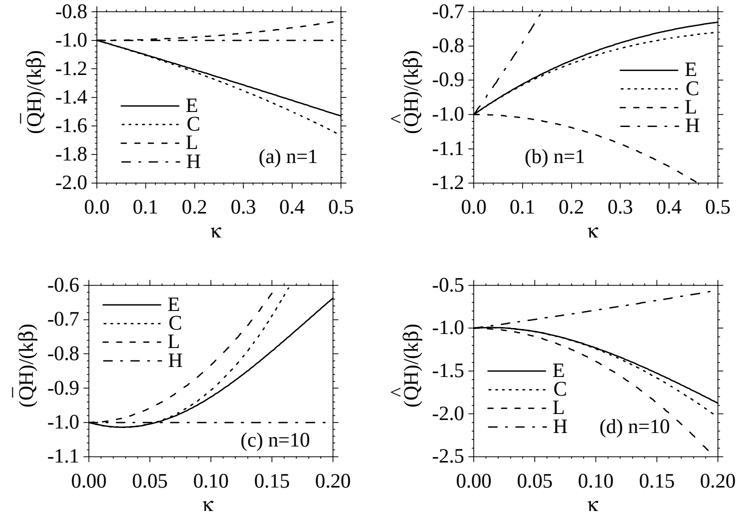

Fig. 2. Response of a spherical shell. Comparison of the exact (E), Cosserat (C), Lukasiewicz (L) and Hashin (H) solutions.

variation of the temperature field through the shell's thickness. Also, it is noted that the Legendre functions for $n = 1$ and $n = 10$ are given by

$$P_1(\mu) = \mu, \quad P_{10}(\mu) = -\frac{63}{256} + \frac{3465}{256}\mu^2 - \frac{15015}{128}\mu^4 + \frac{45045}{128}\mu^6 - \frac{109395}{256}\mu^8 + \frac{46189}{256}\mu^{10}. \tag{10.16}$$

11. Summary

The main idea of this paper is to point out that linear theories for rigid heat conducting shells which introduce expressions for the average temperature $\theta(\theta^\alpha, t)$ and average temperature gradient $\theta_3(\theta^\alpha, t)$ should be capable of reproducing exact solutions for all constant three-dimensional temperature gradients $\mathbf{g}^*$. Specifically, it was shown in Section 4 that, within the context of the direct approach to Cosserat shell theory, the constitutive equations for the entropy fluxes $\{p^\alpha, p^3, p^{3\alpha}\}$ should satisfy the restrictions (4.8) whenever the temperature gradient $\mathbf{g}^*$ is constant and the temperature fields are given by (4.5). In addition, these constitutive equations must satisfy the second law of thermodynamics (3.11) which requires heat to flow from hot to cold regions.

The Bubnov–Galerkin forms for $\{p^\alpha, p^{3\alpha}\}$ can be obtained analytically for general shells but they are quite complicated. In contrast, the expressions (3.12) and (A.10a,b) for $\{p^\alpha, p^{3\alpha}\}$, which are motivated by the Bubnov–Galerkin forms for a plate, are used for general shells. The restriction (4.8a) is satisfied by the constitutive equation (3.12) for $\{p^\alpha\}$, but the restriction (4.8b) is not satisfied when $\{p^{3\alpha}\}$ is given by (3.12) and $\{p^3\}$ is given by the Bubnov–Galerkin form (3.14). Moreover, since $\{p^3, p^{3\alpha}\}$ are two linear functions of three variables $\{\theta_{,\alpha}, \theta_3, \theta_{3,\alpha}\}$ and the shell's geometry, it is clear that the single restriction (4.8b) is not sufficient to uniquely determine these constitutive equations. The development in Section 4 uses the Bubnov–Galerkin form (3.12) for $\{p^{3\alpha}\}$ and writes (4.8b) in the alternative form (4.10). Then, a new constitutive form (4.11), with the specifications (4.13), is proposed for $\{p^3\}$ to satisfy the restriction (4.10) and the second law of thermodynamics.

The resulting set of linear constitutive equations ensures that the Cosserat theory admits exact solutions for all constant temperature gradients $\mathbf{g}^*$ and all shell geometries, including shells with variable thicknesses. Furthermore, the example problems discussed in Sections 8 and 10 demonstrate that the resulting Cosserat theory is reasonably accurate for moderately thick shells and moderately strong variation of the temperature field through the shell's thickness.

Acknowledgement

The author would like to thank Y. Benveniste for drawing his attention back to the importance of heat conduction in rigid shells.

Appendix A. Three-dimensional approach

Multiplying the balance of entropy (2.13a) by a weighting function w and integrating the result over the thickness of the shell leads to

$$\begin{aligned}\int_{-H/2}^{H/2} wm^*\dot{\eta}^*\,\mathrm{d}\theta^3 &= \int_{-H/2}^{H/2}[wm^*s^* + wm^*\xi^* + w_{,3}p^{*3}]\,\mathrm{d}\theta^3 - \left[\left(wp^{*3} - \frac{1}{2}wp^{*\alpha}H_{,\alpha}\right)\Big|_{\theta^3=H/2}\right.\\ &\left.\quad - \left(wp^{*3} + \frac{1}{2}wp^{*\alpha}H_{,\alpha}\right)\Big|_{\theta^3=-H/2}\right] - \left[\int_{-H/2}^{H/2}(wp^{*\alpha})\,\mathrm{d}\theta^3\right]_{,\alpha}. \qquad \text{(A.1)}\end{aligned}$$

Also, integrating the energy equation (2.13b) over the thickness yields

$$\begin{aligned}\int_{-H/2}^{H/2} m^*\dot{\varepsilon}^*\,\mathrm{d}\theta^3 &= \int_{-H/2}^{H/2} m^*\theta^*s^*\,\mathrm{d}\theta^3 - \left[\left(\theta^*p^{*3} - \frac{1}{2}H_{,\alpha}\theta^*p^{*\alpha}\right)\Big|_{\theta^3=H/2}\right.\\ &\left.\quad - \left(\theta^*p^{*3} + \frac{1}{2}H_{,\alpha}\theta^*p^{*\alpha}\right)\Big|_{\theta^3=-H/2}\right] - \left[\int_{-H/2}^{H/2}(\theta^*p^{*\alpha})\,\mathrm{d}\theta^3\right]_{,\alpha}. \qquad \text{(A.2)}\end{aligned}$$

Then, taking $w = 1$ and $w = \theta^3$ and introducing the definitions

$$\begin{aligned}&m = \int_{-H/2}^{H/2} m^*\,\mathrm{d}\theta^3, \quad m\eta = \int_{-H/2}^{H/2} m^*\eta^*\,\mathrm{d}\theta^3, \quad m\eta^3 = \int_{-H/2}^{H/2} \theta^3 m^*\eta^*\,\mathrm{d}\theta^3,\\ &ms = m(s_s + s_p), \quad ms_s = \int_{-H/2}^{H/2} m^*s^*\,\mathrm{d}\theta^3,\\ &ms_p = -\left[\left(\hat{p}^{*3} - \frac{1}{2}H_{,\sigma}\hat{p}^{*\sigma}\right) - \left(\bar{p}^{*3} + \frac{1}{2}H_{,\sigma}\bar{p}^{*\sigma}\right)\right],\\ &ms^3 = m(s_s^3 + s_p^3), \quad ms_s^3 = \int_{-H/2}^{H/2} \theta^3 m^*s^*\,\mathrm{d}\theta^3,\\ &ms_p^3 = -\frac{H}{2}\left[\left(\hat{p}^{*3} - \frac{1}{2}H_{,\sigma}\hat{p}^{*\sigma}\right) + \left(\bar{p}^{*3} + \frac{1}{2}H_{,\sigma}\bar{p}^{*\sigma}\right)\right],\\ &m\xi = \int_{-H/2}^{H/2} m^*\xi^*\,\mathrm{d}\theta^3, \quad m\xi^3 = \int_{-H/2}^{H/2} \theta^3 m^*\xi^*\,\mathrm{d}\theta^3,\\ &p^i = \int_{-H/2}^{H/2} p^{*i}\,\mathrm{d}\theta^3, \quad p^{3\alpha} = \int_{-H/2}^{H/2} \theta^3 p^{*\alpha}\,\mathrm{d}\theta^3, \quad m\varepsilon = \int_{-H/2}^{H/2} m^*\varepsilon^*\,\mathrm{d}\theta^3,\\ &\bar{p}^{*i} = p^{*i}\left(\theta^\alpha, -\frac{H}{2}, t\right), \quad \hat{p}^{*i} = p^{*i}\left(\theta^\alpha, \frac{H}{2}, t\right),\end{aligned} \qquad \text{(A.3)}$$

yields the balances of entropy (3.5) and the balance of energy (3.6).

Next, using the kinematic expression (3.1) with the definitions (3.2) it follows that

$$\begin{aligned}
&\mathbf{g}^*_\alpha = \mathbf{a}_\alpha + \theta^3 \mathbf{a}_{3,\alpha}, \quad \mathbf{g}^*_3 = \mathbf{a}_3, \\
&g^{*1/2}\mathbf{g}^{*1} = a^{1/2}\mathbf{a}^1 + \theta^3(\mathbf{a}_{3,2} \times \mathbf{a}_3), \\
&g^{*1/2}\mathbf{g}^{*2} = a^{1/2}\mathbf{a}^2 + \theta^3(\mathbf{a}_3 \times \mathbf{a}_{3,1}), \\
&\mathbf{g}^{*3} = \mathbf{a}_3, \quad g^{*1/2} = \left[a^{1/2} + \theta^3 a^{1/2}(\mathbf{a}^\sigma \cdot \mathbf{a}_{3,\sigma}) + (\theta^3)^2(\mathbf{a}_{3,1} \times \mathbf{a}_{3,2} \cdot \mathbf{a}_3)\right].
\end{aligned} \tag{A.4}$$

Thus, for constant three-dimensional mass density ρ^* the quantity m in (A.3) is given by

$$m = \rho^* H\left[a^{1/2} + \frac{H^2}{12}(\mathbf{a}_{3,1} \times \mathbf{a}_{3,2} \cdot \mathbf{a}_3)\right]. \tag{A.5}$$

Moreover, using the kinematic assumption (3.1) it follows that the unit outward normal vectors, $\bar{\mathbf{n}}^*$ on the bottom surface and $\hat{\mathbf{n}}^*$ on the top surface become

$$\begin{aligned}
&\bar{\alpha}\mathbf{n}^* = -\left(\mathbf{a}_1 - \frac{H}{2}\mathbf{a}_{3,1} - \frac{H_{,1}}{2}\mathbf{a}_3\right) \times \left(\mathbf{a}_2 - \frac{H}{2}\mathbf{a}_{3,2} - \frac{H_{,2}}{2}\mathbf{a}_3\right) \\
&\bar{\alpha}\bar{\mathbf{n}}^* = -\left[\bar{g}^{*1/2}\bar{\mathbf{g}}^{*3} + \frac{H_{,\sigma}}{2}\bar{g}^{*1/2}\bar{\mathbf{g}}^{*\sigma}\right], \\
&\hat{\alpha}\hat{\mathbf{n}}^* = \left(\mathbf{a}_1 + \frac{H}{2}\mathbf{a}_{3,1} + \frac{H_{,1}}{2}\mathbf{a}_3\right) \times \left(\mathbf{a}_2 + \frac{H}{2}\mathbf{a}_{3,2} + \frac{H_{,2}}{2}\mathbf{a}_3\right) \\
&\hat{\alpha}\hat{\mathbf{n}}^* = \hat{g}^{*1/2}\hat{\mathbf{g}}^{*3} - \frac{H_{,\sigma}}{2}\hat{g}^{*1/2}\hat{\mathbf{g}}^{*\sigma},
\end{aligned} \tag{A.6}$$

where

$$\begin{aligned}
&\bar{g}^{*1/2}\bar{\mathbf{g}}^{*1} = a^{1/2}\mathbf{a}^1 - \frac{H}{2}(\mathbf{a}_{3,2} \times \mathbf{a}_3), \quad \bar{g}^{*1/2}\bar{\mathbf{g}}^{*2} = a^{1/2}\mathbf{a}^2 - \frac{H}{2}(\mathbf{a}_3 \times \mathbf{a}_{3,1}), \\
&\bar{g}^{*1/2}\bar{\mathbf{g}}^{*3} = \left[a^{1/2} - \frac{H}{2}a^{1/2}(\mathbf{a}^\sigma \cdot \mathbf{a}_{3,\sigma}) + \frac{H^2}{4}(\mathbf{a}_{3,1} \times \mathbf{a}_{3,2} \cdot \mathbf{a}_3)\right]\mathbf{a}_3, \\
&\hat{g}^{*1/2}\hat{\mathbf{g}}^{*1} = a^{1/2}\mathbf{a}^1 + \frac{H}{2}(\mathbf{a}_{3,2} \times \mathbf{a}_3), \quad \hat{g}^{*1/2}\hat{\mathbf{g}}^{*2} = a^{1/2}\mathbf{a}^2 + \frac{H}{2}(\mathbf{a}_3 \times \mathbf{a}_{3,1}), \\
&\hat{g}^{*1/2}\hat{\mathbf{g}}^{*3} = \left[a^{1/2} + \frac{H}{2}a^{1/2}(\mathbf{a}^\sigma \cdot \mathbf{a}_{3,\sigma}) + \frac{H^2}{4}(\mathbf{a}_{3,1} \times \mathbf{a}_{3,2} \cdot \mathbf{a}_3)\right]\mathbf{a}_3.
\end{aligned} \tag{A.7}$$

Furthermore, using the relationship (2.15) for $\mathbf{p}^*$ in terms of $\mathbf{q}^*$ it can be seen that the external rates of supply of entropy $\{ms_p, ms_p^3\}$ are given by (3.17).

Next, with the help of (2.11a), (2.12), (2.15), (3.4) and (A.4) it follows that:

$$p^{*\alpha} = -\frac{k}{\theta_0}\left[\frac{1}{g^{1/2}}(g^{*1/2}\mathbf{g}^{*\alpha} \cdot g^{*1/2}\mathbf{g}^{*\beta})(\theta_{,\beta} + \theta^3\theta_{3,\beta})\right], \tag{A.8a}$$

$$p^{*3} = -\frac{k}{\theta_0}\left[a^{1/2} + \theta^3 a^{1/2}(\mathbf{a}^\sigma \cdot \mathbf{a}_{3,\sigma}) + (\theta^3)^2(\mathbf{a}_{3,1} \times \mathbf{a}_{3,2} \cdot \mathbf{a}_3)\right]\theta_3. \tag{A.8b}$$

Then, the constitutive equations for $\{p^i, p^{3\alpha}\}$ associated with the Bubnov–Galerkin procedure are obtained by evaluating the integrals in (A.3) using the approximate expressions (A.8). The resulting expressions for $\{p^\alpha, p^{3\alpha}\}$ can be obtained analytically for a general shell but they are quite complicated. In contrast, the expression for p^3 remains relatively simple and is given by (3.14).

For the case of a flat plate the unit normal $\mathbf{a}_3$ is independent of θ^α so that with the help of (3.1) and (3.2) the kinematic quantities (2.10) reduce to

$$\mathbf{g}_i^* = \mathbf{a}_i, \quad g^{*\alpha\beta} = a^{\alpha\beta}, \quad g^{*3\alpha} = 0, \quad g^{*33} = 1, \quad g^{*1/2} = a^{1/2}. \tag{A.9}$$

Then, the Bubnov–Galerkin forms of the constitutive equations for a plate become

$$p^\alpha = -\frac{kH}{\theta_0} a^{1/2} a^{\alpha\beta} \theta_{,\beta}, \quad p^{3\alpha} = -\frac{kH^3}{12\theta_0} a^{1/2} a^{\alpha\beta} \theta_{3,\beta}, \quad p^3 = -\frac{kH}{\theta_0} a^{1/2} \theta_3. \tag{A.10a,b,c}$$

References

Altay, G.A., Dokmeci, M.C., 2001. Coupled thermoelastic shell equations with second sound for high-frequency vibrations of temperature-dependent materials. Int. J. Solids Struct. 38, 2737–2768.

Boley, B.A., Weiner, J.H., 1960. Theory of Thermal Stresses. Wiley, New York.

Carslaw, H.S., Jaeger, J.C., 1956. Conduction of heat in solids. Oxford University Press, London.

Green, A.E., Naghdi, P.M., 1977. On thermodynamics and the nature of the second law. Proc. R. Soc. London A 357, 253–270.

Green, A.E., Naghdi, P.M., 1978. The second law of thermodynamics and cyclic processes. J. Appl. Mech. 45, 487–492.

Green, A.E., Naghdi, P.M., 1979. On thermal effects in the theory of shells. Proc. R. Soc. London A 365, 161–190, A367, 572.

Green, A.E., Zerna, W., 1968. Theoretical Elasticity. Oxford University Press, Oxford.

Hashin, Z., 2001. Thin interphase/imperfect interface in conduction. J. Appl. Phys. 89, 2261–2267.

Hetnarski, R.B. (Ed.), 1986. Thermal Stresses, vol. I. North-Holland, New York.

Hetnarski, R.B. (Ed.), 1987. Thermal Stresses, vol. II. North-Holland, New York.

Hetnarski, R.B. (Ed.), 1989. Thermal Stresses, vol. III. North-Holland, New York.

Hetnarski, R.B. (Ed.), 1996. Thermal Stresses, vol. IV. Elseiver, New York.

Lukasiewicz, S.A., 1989. Thermal stresses in shells. In: Hetnarski, R.B. (Ed.), Thermal Stresses, vol. III. North-Holland, New York.

Naghdi, P.M., 1972. The theory of shells and plates. In: Truesdall, C. (Ed.), S. Flugge's Handbuch der Physik, vol. VIa/2. Springer-Verlag, Berlin, pp. 425–640.

Naghdi, P.M., Rubin, M.B., 1995. Restrictions on nonlinear constitutive equations for elastic shells. J. Elast. 39, 133–163.

Pham Huy, H., Sanchez-Palencia, E., 1974. Phénomènes de transmission a travers des couches minces de conductivité élevée. J. Math. Anal. Appl. 47, 284–309.

Rubin, M.B., 1986. Heat conduction in plates and shells with emphasis on a conical shell. Int. J. Solids Struct. 22, 527–551.

Rubin, M.B., 2000. Cosserat Theories: Shells, Rods and Points. In: Solid Mechanics and its Applications, vol. 79. Kluwer, The Netherlands.

Sanchez-Palencia, E., 1970. Comportement limite d'un problème de transmission a travers une plaque faiblement conductrice. Comp. Rend. Acad. Sci. Paris Ser. A270, 1026–1028.

Shvets, R.M., Flyachok, V.M., 1999. Heat conduction equations for multilayer anisotropic shells. J. Therm. Stress. 22, 241–254.